SYSTEMS
RESEARCH
CENTER

Geometry, Dynamics and Control
of Coupled Systems

by L.S. Wang
Advisor: P.S. Krishnaprasad

1. REPORT DATE **1990**	2. REPORT TYPE	3. DATES COVERED **00-00-1990 to 00-00-1990**
4. TITLE AND SUBTITLE **Geometry, Dynamics and Control of Coupled Systems**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Maryland,The Graduate School,2123 Lee Building,College Park,MD,20742**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES **209**	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**			

GEOMETRY, DYNAMICS AND CONTROL
OF COUPLED SYSTEMS

By

Li-Sheng Wang

Dissertation submitted to the faculty of the Graduate School
of University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1990

Advisory Committee :

Professor P. S. Krishnaprasad, Chairman/Advisor
Professor S. S. Antman
Professor A. L. Tits
Associate Professor J. H. Maddocks
Assistant Professor W. P. Dayawansa

ABSTRACT

Title of Dissertation: Geometry, Dynamics and Control of Coupled Systems

Li-Sheng Wang,   Doctor of Philosophy, 1990

Dissertation directed by: Dr. P. S. Krishnaprasad
Professor
Department of Electrical Engineering

In this dissertation, we study the dynamics and control of coupled mechanical systems. A key feature of this work is the systematic use of modern geometric mechanics, including methods based on symplectic geometry, Lie symmetry groups, reductions, lagrangian mechanics and hamiltonian mechanics to investigate specific Eulerian many-body problems. A general framework for gyroscopic systems with symmetry is introduced and analyzed. The influence of the gyroscopic term (linear term in Lagrangian) on the dynamical behavior is exploited. The notion of gyroscopic control is proposed to emphasize the role of the gyroscopic term in designing control algorithms. The block-diagonalization techniques associated to the energy-momentum method which proved to be very useful in determining stability for simple mechanical systems with symmetry are successfully extended to gyroscopic systems with symmetry. The techniques developed here are applied to several interesting mechanical systems. These examples include the dual-spin method of attitude control for artificial satellites, a multi-body analog of the dual-spin problem, a rigid body with momentum wheels in a central gravitational force field, and a rigid body with momentum wheels and a flexible attachment.

ACKNOWLEDGEMENTS

Finally, I deeply thank my family, especially my mother, Ping-Ching, and my wife, Yuh-Yin, for their continuous understanding, enduring my random timetable, and moral as well as living support. I also like to thank my one-and-a-half year-old daughter, Iru, whose birth further stimulated me to complete this dissertation in a timely manner.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

Introduction

Coupled systems very often arise in the design and control of complicated mechanical systems, such as robots and complex spacecraft. The kinematics and dynamics of each individual body in such a system are highly coupled with the motions of the other bodies. Geometry, dynamics and control are three essential components of a rigorous study of the problems underlying such systems. A rational approach is to first recognize the geometrical structure of the system, followed by a study of the dynamical behavior based on fundamental principles. With this understanding, control algorithms can be then explored to fulfill design requirements.

This dissertation is part of an on-going program to understand the dynamics and control of multibody systems from a modern point of view, see [65] [54]. In [37], questions concerning the dynamics of systems of kinematically coupled rigid or flexible bodies are referred to as Eulerian many-body problems to emphasize the role of Euler forces (or frame forces) in determining the nature of inter-body interactions. Typical spacecraft designs, such as that of the Explorer, Orbiting Geophysical Observatory, Hubble Space Telescope, etc., present examples of such problems. The Remote Manipulator System on the space shuttle, the proposed Tele-robot Work System, and many terrestrial robots are further instances. A careful investigation and understanding of such problems is necessary to achieve successful design and control of such complicated systems with demanding mission objectives.

The key feature of this dissertation is the systematic use of modern geometric mechanics, including methods based on symplectic geometry, Lie symmetry groups, reductions, lagrangian mechanics and hamiltonian mechanics, to explore specific Eulerian many-body problems. This intrinsic approach provides us with a tool which is global and coordinate free; thus avoiding the singularity problems and cumbersome calcula-

1

tions associated with local coordinates, e.g. Euler angles for the rotation group $SO(3)$.
In the following, Chapter Two describes the invariant form of the Lagrange-d'Alembert
Principle in lagrangian mechanics. The special geometric structure of the rotation group
or the special orthogonal group $SO(3)$ is exploited and used to display the classical
Euler-Lagrange equations in intrinsic geometric form. This development provides funda-
mental connections between lagrangian mechanics and hamiltonian mechanics whenever
problems arise with the rotation group appearing as a factor in the configuration space.
Chapter Three introduces the general framework of gyroscopic systems with symmetry.
This framework plays a central role in the following development. Reduction in both
symplectic and Poisson senses are worked out in detail. One variational characteriza-
tion, namely the principle of symmetric criticality, for the relative equilibria is derived
for gyroscopic systems and applied to a mechanical system consisting of two rigid bodies
connected by a ball-in-socket joint.

In designing a communication satellite or an interplanetary probe, engineers are
often faced with the requirement that the spacecraft be able to maintain a fixed ori-
entation relative to some inertial frame. In the process of attitude acquisition, damp-
ing effects play an important role. Chapter Four initiates a discussion of hamiltonian
systems with some added dissipation. A multibody analog of the classical dual-spin
problem is formulated and an asymptotic stability theorem is established. Namely, the
system asymptotically approaches a stable relative equilibrium of a suitable limiting
gyroscopic system. This leads us to consider stability issues. Accordingly, several tech-
niques in stability analysis are discussed in Chapter Five. The energy-Casimir method,
the Lagrange-multiplier method, and the energy-momentum method are discussed. In
particular, the block diagonalization technique arising in the energy-momentum method
for simple mechanical systems with symmetry [62] [61] is extended to gyroscopic systems
with symmetry.

In the study of Newtonian many-body problems in celestial mechanics, it is
customary to treat the bodies as point masses. See, for example, Sternberg [68],
Smale [64], and Abraham and Marsden [2]. However the proper accounting of stable
planetary spins for instance, would seem to require the consideration of bodies of finite

2

extent. This makes the problem an Eulerian many-body problem. In Chapter Six, the hamiltonian framework of a rigid body in a central gravitational force field is formulated. The reduced hamiltonian formulation introduced provides a systematic approach to the approximation of the underlying dynamics based on a series expansion of the reduced Hamiltonian. These approximations preserve the structure of the system. The energy-Casimir method and the Lagrange-multiplier method are then used to prove stability results.

The main feature of *gyroscopic* systems with symmetry resides in the linear term in the Lagrangian. The essence of *gyroscopic control* is to investigate the role of that linear term in dynamical behavior, and furthermore to utilize that linear term to control the system dynamics. These issues are illustrated in great detail in Chapter Seven with four examples. They are dual-spin, dual-spin in a central gravitational force field, multibody dual-spin, dual-spin with a flexible attachment. The last example is an infinite-dimensional system and the modeling techniques developed in Chapter Two are extended to such cases. The effects of the linear term on stability are explored for each problem by using the techniques developed in Chapter Five. The discussions shed some light on a general methodology in attitude control.

This dissertation concludes with Chapter Eight where some future directions for research are also outlined.

There are three numbering systems in this dissertation. One is for equations, one is for Figures and Tables, the other is for Definitions, Theorems, Lemmas, Corollaries, Propositions, and Remarks. They all have three digits, with the first and second digit representing the chapter and section number in which they occur in, respectively. The third one is the order of occurrence of that item in the corresponding section.

CHAPTER II

Lagrangian Mechanics

Lagrangian mechanics provides a systematic way of dealing with mechanical problems from a unified point of view. In contrast with working on the cotangent bundle as in most of hamiltonian mechanics, lagrangian mechanics formulates the problems on the tangent bundle, or the coordinate-velocity space. As we shall see, it admits greater freedom in interpreting and formulating intuitive physical notions such as forces and the principle of virtual power. Especially for nonholonomic mechanical systems, there is no nice "hamiltonian", or "symplectic" equivalent notion to lagrangian mechanics. In this chapter, we discuss the lagrangian formalism in its invariant form, and then apply it to the dynamical modeling of some specific problems. A key source of inspiration is [69].

2.1. Intrinsic Form

In this section, we introduce the invariant form of lagrangian mechanics through local representations and show that the invariant form of the Lagrange-d'Alembert Principle gives rise to the Euler-Lagrange Equations in local coordinates.

Let Q be a smooth manifold (configuration space) with local coordinates x, TQ be the tangent bundle of Q with local coordinates (x, v), $\pi : TQ \rightarrow Q$ be the canonical projection defined by

$$\pi(x, v) \stackrel{\triangle}{=} x.$$

The tangent map $T\pi$ of the canonical projection, $T\pi : T(TQ) \rightarrow TQ$, can be expressed in local coordinates as

$$T\pi_{(x,v)}(u, w) = u \quad \in T_x Q,$$

4

which is a projection from $T(TQ)$ to TQ. Since the Euler-Lagrange Equations are second-order differential equations, we need to consider the corresponding elements in the *jet spaces* of Q, namely the second tangent vectors. Let $T(TQ)$ be the second tangent bundle with local coordinates $(x,\ v,\ u,\ w)$. Let $T_{(x,v)}$ be the tangent space of TQ at $(x,\ v)$, i.e. $T_{(x,v)}TQ$. Let $T^V_{(x,v)}$ denote the *vertical* tangent subspace of $T_{(x,v)}$ which consists of vectors tangent to the fiber of TQ. In local coordinates, each vector in $T^V_{(x,v)}$ can be written as $(0,\ w)$, for some $w \in T_xQ$.

Define the map

$$\gamma_{(x,v)} : T_xQ \ \rightarrow \ T_{(x,v)}\,TQ$$
$$\gamma_{(x,v)} \cdot u \ \triangleq \ (0,\ u) \qquad \in \ T^V_{(x,v)} \tag{2.1.1}$$

which is an isomorphism between T_xQ and $T^V_{(x,v)}$. Let $X\ :\ TQ\ \rightarrow\ T(TQ)$ be a vector field on TQ. X is called a *vertical vector field* if $X(x,\ v) \in T^V_{(x,v)}$ or

$$X(x,\ v) = (0,\ w), \quad \text{for some } w \ \in \ T_xQ.$$

This is equivalent to saying $T\pi \cdot X\ =\ 0$. A vector field X^{PV} is called the *principal vertical field* if it is a vertical vector field and

$$X^{PV}(x,\ v) \ = \ \gamma_{(x,v)} \cdot v,$$

which is $(0,\ v)$ in local coordinates. Let $\ \omega\ $ be a 1-form on TQ. It is said to be *horizontal* if for all vertical vector fields $\ X$, $\omega(X) = 0$, or in local coordinates,

$$\omega(x,\ v) \ = \ (\alpha,\ 0), \quad \text{for some } \alpha \ \in \ T^*_xQ.$$

The *dual* of the map $T\pi_{(x,v)}\ :\ T_{(x,v)}\,TQ\ \rightarrow\ T_x\,Q$ can be defined as

$$T\pi^*_{(x,v)} :\ T^*_xQ\ \rightarrow\ T^*_{(x,v)}TQ$$
$$\left(T\pi^*_{(x,v)}\alpha\right)(u,w) \ = \ \alpha\,\left(\,T\pi_{(x,v)}\,(u,w)\right) \ = \ \alpha(u)$$

where $\alpha \in T^*_xQ$. Thus, $T\pi^*_{(x,v)}\ \alpha\ = (\alpha,0)$, in local coordinates. Also, we denote the dual of $\gamma_{(x,v)}$,

$$\gamma^*_{(x,v)} :\ T^*_{(x,v)}\,TQ\ \rightarrow\ T^*_xQ$$

5

in local coordinates,

$$\left(\gamma^*_{(x,v)}(\alpha, \ \beta)\right)(u) = (\alpha, \ \beta)\left(\gamma_{(x,v)} \cdot u\right)$$

$$= (\alpha, \ \beta)\,(0, \ u)$$

$$= \beta(u).$$

Equivalently, $\gamma^*_{(x,v)}(\alpha, \ \beta) = \beta$. Now we define the map $\tau : T^*(TQ) \to T^*(TQ)$ to be

$$\tau_{(x,v)} = T\pi^*_{(x,v)} \cdot \gamma^*_{(x,v)}. \qquad (2.1.2)$$

In particular, for $(\alpha, \beta) \in T^*_{(x,v)}TQ$,

$$\tau_{(x,v)} \ (\alpha, \ \beta) = T\pi^*_{(x,v)} \cdot \gamma^*_{(x,v)} \ (\alpha, \ \beta)$$

$$= T\pi^*_{(x,v)} \cdot \beta$$

$$= (\beta, \ 0).$$

Thus, we know that $\tau_{(x,v)}$ maps any cotangent vector (*covector*) to a horizontal covector by the mapping

$$(\alpha, \ \beta) \ \mapsto \ (\beta, \ 0).$$

Globally τ maps any 1-form on TQ to a horizontal 1-form on TQ. On the other hand, we may define a map from the second tangent bundle to itself

$$\tau_* \ : \ T(TQ) \ \to \ T(TQ)$$

as

$$\tau_{*(x,v)} \ \stackrel{\triangle}{=} \ \gamma_{(x,v)} \cdot T\pi_{(x,v)}. \qquad (2.1.3)$$

In local coordinates we can associate to each $(u, w) \in T_{(x,v)}TQ$,

$$\tau_{*(x,v)} \ (u, \ w) \ = \ \gamma_{(x,v)} \cdot T\pi_{(x,v)} \ (u, \ w)$$

$$= \ \gamma_{(x,v)} \ (u)$$

$$= (0, \ u).$$

6

In other words, $\tau_{*(x,v)}$ maps any second tangent vector to a vertical tangent vector by the map

$$(u,\ w) \mapsto (0,\ u),$$

and τ_* maps a vector field on TQ to a vertical vector field on TQ.

DEFINITION 2.1.1

$X \in \mathfrak{X}(TQ)$, a vector field on TQ, is a *special vector field* if and only if

$$\tau_* X = X^{PV}.$$

∎

In local coordinates, assuming $X(x,v) = (u,w)$, it says

$$\tau_* X(x,v) = \tau_*(u,\ w) = (0,\ u) = X^{PV}(x,\ v) = (0,\ v),$$

which is equivalent to the condition $u = v$. We thus know that this definition of special vector field is the same as saying X gives rise to a second-order equation on Q in the sense of Abraham and Marsden [2], or $T\pi \cdot X = identity$ on TQ.

Now, let $L : TQ \to \mathbb{R}$ be a smooth function (called *Lagrangian*). The corresponding differential 1-form

$$dL \ : \ TQ \to \ T^*(TQ)$$

can be written in local coordinates as,

$$\begin{aligned}
dL(x,v)\,(u,\ w) &= TL_{(x,v)} \cdot (u,\ w) \\
&= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L(x + \epsilon u,\ v + \epsilon w) \qquad\qquad (2.1.4)\\
&= D_1 L(x,v) \cdot u \ + \ D_2 L(x,v) \cdot w.
\end{aligned}$$

where $(x + \epsilon u,\ v + \epsilon w)$ denotes a curve in TQ generated by the tangent vector $(u,w) \in T_{(x,v)}TQ$. Recall that tangent vectors may be defined through the equivalence classes of curves. For the tangent vector (u,w), the curve $(x + \epsilon u,\ v + \epsilon w)$ is thus an

7

element in the equivalent class of curves associated with (u, w). In the following, we sometimes write

$$(u,\ w)\ =\ [\ (x + \epsilon\, u,\ v + \epsilon\, w)\ ].$$

From (2.1.4), we have the expression

$$dL(x,v)\ =\ (D_1 L(x,v),\ D_2 L(x,v)).$$

The horizontal 1-form Θ_L on TQ corresponding to L is defined to be

$$\Theta_L\ \triangleq\ \tau \cdot dL. \tag{2.1.5}$$

In local coordinates,

$$\begin{aligned}
\Theta_L(x,v)\ &=\ \tau_{(x,v)} \cdot dL(x,v) \\
&=\ \tau_{(x,v)} \cdot (D_1 L(x,v),\ D_2 L(x,v)) \\
&=\ (D_2 L(x,v),\ 0).
\end{aligned}$$

Taking the exterior derivative, we can associate to L a 2-form on TQ

$$\Omega_L\ :\ T(TQ)\ \times\ T(TQ) \to \mathbb{R}$$

$$\Omega_L\ \triangleq\ -\, d\, \Theta_L. \tag{2.1.6a}$$

If we write

$$\Theta_L\ =\ D_{v^i} L\ dx^i,$$

then, by taking exterior differentiations on both sides, we get

$$\begin{aligned}
\Omega_L\ &=\ -d D_{v^i} L\ \wedge\ dx^i \\
&=\ -(D_{x^j} D_{v^i} L\ dx^j\ +\ D_{v^j} D_{v^i} L\ dv^j)\ \wedge\ dx^i \\
&=\ D_{x^j} D_{v^i} L\ dx^i\ \wedge\ dx^j\ +\ D_{v^j} D_{v^i} L\ dx^i\ \wedge\ dv^j. \tag{2.1.6b}
\end{aligned}$$

We have another expression for the 2-form Ω_L. Letting (u_1, w_1), $(u_2, w_2) \in T_{(x,v)} TQ$, we have the following formula in local coordinates,

8

$$\Omega_L(x,v)\big((u_1, \ w_1), \ (u_2, \ w_2)\big)$$

$$= \ - D\Theta_L(x,v)(u_1, \ w_1) \cdot (u_2, \ w_2) \ + \ D\Theta_L(x,v)(u_2, \ w_2) \cdot (u_1, \ w_1)$$

$$= (D_1 D_2 L(x,v) \cdot u_2) \cdot u_1 \ + \ (D_2 D_2 L(x,v) \cdot w_2) \cdot u_1$$

$$- \ (D_1 D_2 L(x,v) \cdot u_1) \cdot u_2 \ - \ (D_2 D_2 L(x,v) \cdot w_1) \cdot u_2 \qquad (2.1.6c)$$

Note that this formula is exactly the same as the Lagrange 2-form ω_L in p.211 of [2], which is derived from the canonical symplectic 2-form on the cotangent bundle T^*Q. Here we proceed in a direct way.

Let $T^{*H}_{(x,v)}$ denote the space of horizontal covectors at $(x, \ v)$ in TQ. Define the map

$$\sigma \ : \ T^{*H}_{(x,v)} \ \to \ T^*_x Q$$

to be, in local coordinates, for $\alpha \in T^*_x Q$,

$$\sigma_{(x,v)}(\alpha, \ 0) \ \overset{\triangle}{=} \ \alpha.$$

Next, we give the intrinsic form of *Legendre transformation* which maps the velocity phase space to the momentum phase space. The Legendre transformation corresponding to the Lagrangian L can be defined as

$$\ell_L \ : \ TQ \ \to \ T^*Q$$
$$(x, \ v) \ \mapsto \ \big(x, \ \sigma_{(x,v)} \cdot \Theta_L(x,v)\big), \qquad (2.1.7a)$$

or, equivalently,

$$\ell_L(x, \ v) \ = \ (x, \ D_2 L(x,v)) \qquad (2.1.7b)$$

which is exactly the same as defined through fiber derivatives in [2].

Assuming now that ℓ_L is a diffeomorphism (or L is *hyperregular*), we have

9

$$\ell_L^{-1} \ : \ T^*Q \ \rightarrow \ TQ.$$

(This condition implies that, in local coordinates, $D_2 D_2 L(x, v)$ is nonsingular.) Denote the space of k-forms on a manifold M as $\varpi^k(M)$. By the pull-back of ℓ_L^{-1},

$$(\ell_L^{-1})^* \ : \ \varpi^2(TQ) \ \rightarrow \ \varpi^2(T^*Q),$$

we can define a 2-form on T^*Q as

$$\omega_0 \ \triangleq \ (\ell_L^{-1})^* \Omega_L. \tag{2.1.8a}$$

Although Ω_L is L-dependent, ω_0 defined above is invariant under the change of L. In fact, it is the canonical symplectic 2-form on the cotangent bundle as established by the following lemma.

LEMMA 2.1.2

Letting (x, p) be local coordinates of T^*Q, we have

$$\omega_0 \ = \ dx \ \wedge \ dp, \tag{2.1.8b}$$

which is the canonical symplectic form on the cotangent bundle.

Proof

Let $(v_1, \ \beta_1), \ (v_2, \ \beta_2) \in T_{(x,p)}(T^*Q)$. We compute ω_0 as follows.

$$\omega_0(x, \ p)((v_1, \ \beta_1), \ (v_2, \ \beta_2))$$

$$= \ ((\ell_L^{-1})^* \ \Omega_L)(x, \ p)((v_1, \ \beta_1), \ (v_2, \ \beta_2))$$

$$= \ \Omega_L(\ \ell_L^{-1}(x, \ p))\ (\ T\ell_L^{-1}(v_1, \ \beta_1), \ T\ell_L^{-1}(v_2, \ \beta_2)).$$

Since $T(\ell_L^{-1}) = (T\ell_L)^{-1}$, assuming $\ell_L^{-1}(x, \ p) \ = \ (x, \ v)$, we have

$$T\ell_L^{-1}(v_1, \ \beta_1) \ = \ (u_1, \ w_1) \in T_{(x,v)}TQ$$

$$T\ell_L^{-1}(v_2, \ \beta_2) \ = \ (u_2, \ w_2) \in T_{(x,v)}TQ.$$

Then,

$$
\begin{aligned}
(v_1, \ \beta_1) &= T\ell_L(u_1, \ w_1) \\
&= [\ \ell_L(x \ + \ tu_1, \ v \ + \ tw_1) \ ] \\
&= [\ (x \ + \ tu_1, \ D_2L(x \ + \ tu_1, \ v \ + \ tw_1) \ ) \ ] \\
&= (\ u_1, \ D_1D_2L(x,v) \cdot u_1 \ + \ D_2D_2L(x,v) \cdot w_1),
\end{aligned}
\tag{2.1.9}
$$

which implies

$$
v_1 \ = \ u_1, \qquad \beta_1 \ = \ D_1D_2L(x,v) \cdot u_1 \ + \ D_2D_2L(x,v) \cdot w_1.
$$

Similarly, we have

$$
v_2 \ = \ u_2, \qquad \beta_2 \ = \ D_1D_2L(x,v) \cdot u_2 \ + \ D_2D_2L(x,v) \cdot w_2.
$$

Thus, by the formula of Ω_L in (2.1.6c), we have

$$
\begin{aligned}
\omega_0(x, \ p)&((v_1, \ \beta_1), \ (v_2, \ \beta_2)) \\
&= \Omega_L(x, \ v) \ ((u_1, \ w_1), \ (u_2, \ w_2)) \\
&= - \ (D_1D_2L(x,v) \cdot u_1) \cdot u_2 \ - \ (D_2D_2L(x,v) \cdot w_1) \cdot u_2 \\
&\quad + \ (D_1D_2L(x,v) \cdot u_2) \cdot u_1 \ + \ (D_2D_2L(x,v) \cdot w_2) \cdot u_1 \\
&= \beta_2(u_1) \ - \ \beta_1(u_2)
\end{aligned}
$$

In terms of the wedge product, we obtain ω_0 in the desired form.

$\blacksquare$

From the above discussions, we conclude that when the Legendre transformation is diffeomorphic then the two approaches, either from the cotangent bundle or directly from the tangent bundle are equivalent. Moreover, $(T^*Q, \ \omega_0)$, $(TQ, \ \Omega_L)$ are both symplectic manifolds. From them, we may define corresponding Poisson structures on each manifold.

REMARK 2.1.3

The 2-form Ω_L defined in (2.1.6a) is valid for every Lagrangian L. However, it is nondegenerate, and therefore a symplectic structure, only when L is *regular*, or ℓ_L is a local diffeomorphism. For a *singular* or irregular L, Ω_L becomes *presymplectic*, namely

Ω_L is no longer of maximal rank. Discussions for this case may be found in, e.g. [24] [25].

∎

By using the symplectic 2-form Ω_L, we can define a one-to-one correspondence between the vector fields and 1-forms, $\Pi_L : \varpi^1(TQ) \to \mathfrak{X}(TQ)$ through, for $\omega \in \varpi^1(TQ)$,

$$\Omega_L\big(\Pi_L(\omega), Y\big) = \omega(Y), \quad \forall Y \in \mathfrak{X}(TQ). \tag{2.1.10}$$

In terms of the inverse of Π_L, the expression is

$$\Omega_L(X, Y) = \Pi_L^{-1}(X)(Y), \quad \forall Y \in \mathfrak{X}(TQ),$$

or

$$\Pi_L^{-1}(X)(\cdot) = \Omega_L(X, \cdot).$$

LEMMA 2.1.4

Π_L maps horizontal 1-forms to vertical vector fields.

Proof

We prove the result using local coordinates, and leave reader to find the invariant proof [69]. Let $\omega(x, v) = (\alpha, 0)$. Assume that

$$\Pi_L(\omega)(x, v) = (u, w),$$

where $u, w \in T_x Q$. From (2.1.6c),(2.1.10), we have the following

$$(D_1 D_2 L(x, v) \cdot Y_1) \cdot u + (D_2 D_2 L(x, v) \cdot Y_2) \cdot u$$

$$- (D_1 D_2 L(x, v) \cdot u) \cdot Y_1 - (D_2 D_2 L(x, v) \cdot w) \cdot Y_2 = \langle \alpha, Y_1 \rangle,$$

for all $Y_1, Y_2 \in T_x Q$. Setting $Y_1 = 0$, we get

$$(D_2 D_2 L(x, v) \cdot Y_2) \cdot u = 0, \quad \forall Y_2 \in T_x Q,$$

which implies that u must be zero, and thus $\Pi_L(\omega)$ is a vertical vector field.

∎

12

Now we define the *energy* function on TQ, $H_L : TQ \rightarrow \mathbb{R}$, as

$$H_L = dL(X^{PV}) - L. \qquad (2.1.11a)$$

In local coordinates,

$$
\begin{aligned}
H_L(x, v) &= dL(x, v)(0, v) - L(x, v) \\
&= D_2 L(x, v) \cdot v - L(x, v) \qquad (2.1.11b) \\
&= \ell_L(x, v) \cdot v - L(x, v)
\end{aligned}
$$

which is exactly the same notion as the energy defined in p.213 in [2]. The function $dL(X^{PV})$ is called the *action* corresponding to L in [2]. From the energy function H_L on the velocity space, we may define a function on the momentum phase space as

$$H : T^*Q \rightarrow \mathbb{R},$$

$$H = H_L \circ \ell_L^{-1}.$$

This hamiltonian system (T^*Q, ω_0, H) is the customary object of study in hamiltonian mechanics.

The *Lagrangian vector field* determined by L is defined as,

$$X_{H_L} \triangleq \Pi_L(dH_L), \qquad (2.1.12a)$$

or, equivalently,

$$\Omega_L(X_{H_L}, Z) = dH_L(Z) \quad \forall Z \in \mathfrak{X}(TQ). \qquad (2.1.12b)$$

In local coordinates, the matrix form of (2.1.12b) is

$$X_{H_L}^T [\Omega_L] Z = \nabla H_L^T Z.$$

Thus we may write the Lagrangian vector field as

$$X_{H_L} = \left([\Omega_L]^{-1}\right)^T \nabla H_L. \qquad (2.1.12c)$$

In the language of Poisson structures, X_{H_L} is the Hamiltonian vector field corresponding to the Hamiltonian H_L, and thus H_L is a first integral (conserved quantity) along the

13

vector field X_{H_L}. We say that we can *define consistent equations of motion* if such an X_{H_L} exists.

LEMMA 2.1.5

X_{H_L} is a special vector field.

Proof

Let $X_{H_L}(x, v) = (u_1, w_1)$. By definition (2.1.12b), for all $(u_2, w_2) \in T_{(x,v)}TQ$,

$$\Omega_L(x, v)\left((u_1, w_1), (u_2, w_2)\right) = dH_L(x, v)(u_2, w_2)$$

The RHS (Right Hand Side) of the above equation can be written as

$$\begin{aligned} &dH_L(x, v)(u_2, w_2) \\ &\quad = [\ H_L(x + \epsilon u_2, v + \epsilon w_2)\ ] \\ &\quad = [\ D_2 L(x + \epsilon u_2, v + \epsilon w_2) \cdot (v + \epsilon w_2) - L(x + \epsilon u_2, v + \epsilon w_2)\ ] \\ &\quad = (D_1 D_2 L(x,v) \cdot u_2) \cdot v + (D_2 D_2 L(x,v) \cdot w_2) \cdot v - D_1 L(x,v) \cdot u_2. \end{aligned}$$

By comparing this with (2.1.6c), it can be seen that, for $u_2 = 0$,

$$(D_2 D_2 L(x,v) \cdot w_2) \cdot u_1 = (D_2 D_2 L(x,v) \cdot w_2) \cdot v, \quad \forall\ w_2 \in T_x Q,$$

which implies $u_1 = v$. Thus, X_{H_L} is special.

∎

Now, we are ready to describe the Lagrange-d'Alembert principle. First, we note that for a mechanical system, virtual displacements could be thought as special vector fields on TQ, and forces could be modeled as horizontal 1-forms on TQ. This will be discussed further in the following chapters.

DEFINITION 2.1.6

For a Lagrangian L, the associated *Lagrangian force* on a virtual displacement X, $F_L(X)$, is defined through

$$F_L(X)(Y) \triangleq \Omega_L(X,Y) - dH_L(Y), \quad \forall\, Y \in \mathfrak{X}(TQ) \qquad (2.1.13)$$

■

The Lagrangian force $F_L(X)$ is a 1-form on TQ. For it to be a well-defined *force*, we need the following lemma.

LEMMA 2.1.7

    $F_L(X)$ is a horizontal 1-form on TQ.

Proof

    Since X is a special vector field (the same notion as a virtual displacement), it can be written as

$$X(x,\ v) \ = \ (v,\ w), \text{ where } w \in T_x Q.$$

Letting $(u_2, w_2) \in T_{(x,v)}TQ$, we have, from (2.1.13),

$$F_L(X)(x,\ v)(u_2, w_2)$$

$$= \Omega_L(x,\ v)\left((v,\ w),\ (u_2,\ w_2)\right) - dH_L(x,\ v)(u_2,\ w_2)$$

$$= (D_1 D_2 L(x,v) \cdot u_2) \cdot v + (D_2 D_2 L(x,v) \cdot w_2) \cdot v$$

$$- (D_1 D_2 L(x,v) \cdot v) \cdot u_2 - (D_2 D_2 L(x,v) \cdot w) \cdot u_2$$

$$- (D_1 D_2 L(x,v) \cdot u_2) \cdot v - (D_2 D_2 L(x,v) \cdot w_2) \cdot v + D_1 L(x,v) \cdot u_2$$

$$= -(D_1 D_2 L(x,v) \cdot v) \cdot u_2 - (D_2 D_2 L(x,v) \cdot w) \cdot u_2$$

$$+ D_1 L(x,v) \cdot u_2.$$

For arbitrary vertical vector field Y, $u_2 = 0$, and hence $F_L(X)\,(Y)\ = 0$. Thus the 1-form $F_L(X)$ is horizontal and has the expression, with $X(x,v) = (v,w)$,

$$F_L(X)(x,v)\,(u, w_2) \ = \ (-D_1 D_2 L(x,v) \cdot v - D_2 D_2 L(x,\ v) \cdot w + D_1 L(x,v)\,) \cdot u.$$

■

REMARK 2.1.8

Definition 2.1.6 holds even for L singular, cf. Remark 2.1.3. If L is hyperregular where the Legendre transformation is invertible, we may write, cf. (2.1.10),

$$F_L(X) = \Pi_L^{-1}(X) - dH_L.$$

This is the definition used in [69] for the Lagrangian force.

■

We are ready to state the following principle.

LAGRANGE D'ALEMBERT PRINCIPLE 2.1.9

For a holonomic mechanical system, on the virtual displacement (special vector field) that determines the real trajectory of motion, the sum of the Lagrangian force and the exterior force is 0.

■

In classical mechanics, the Lagrangian force consists of resultant *force of inertia* and forces coming from the potential energy. Thus the principle here corresponds to the classical d'Alembert principle, see e.g. [39]. As discussed in [39], the fundamental entity in analytical mechanics is the virtual work, instead of the classical notion of force. Here we unify the notion in terms of horizontal 1-form, where the *classical forces* are represented by the coordinates of this 1-form.

Let ω be an exterior force or a horizontal 1-form. Principle 2.1.9 says that

$$F_L(X) + \omega = 0, \tag{2.1.14}$$

where X is a special vector field. The trajectories of motion of the mechanical system with Lagrangian L follow the flows of this vector field. In the absence of any exterior force and L being regular, from (2.1.13), we may write

$$\Omega_L(X, Y) = dH_L(Y), \quad \forall Y \in \mathfrak{X}(TQ),$$

which, by definition of X_{H_L}, says that

$$X = X_{H_L},$$

i.e. the Lagrangian vector field gives the real trajectories of motion.

Now we state the Principle 2.1.9 in local coordinates. In local coordinates,

$$F_L(X)(x, v) + \omega(x, v) = 0.$$

Letting $\omega = (\alpha, 0)$, $X(x, v) = (v, w)$, we have

$$\left(-D_1 D_2 L(x, v) \cdot v - D_2 D_2 L(x, v) \cdot w + D_1 L(x, v)\right) \cdot u + \alpha \cdot u = 0, \quad \forall u \in T_x Q.$$

By including time derivatives as $v = \dot{x}$, $w = \dot{v}$, we get

$$\frac{d}{dt} D_2 L(x, v) \cdot u = D_1 L(x, v) \cdot u + \alpha \cdot u, \qquad \forall u \in T_x Q. \tag{2.1.15}$$

Integrating both sides with respect to the variable t, the equation can be rewritten as

$$D_2 L(x, v) \cdot u \Big|_0^T - \int_0^T D_2 L(x, v) \cdot u_t \, dt = \int_0^T \left(D_1 L(x, v) \cdot u + \alpha \cdot u\right) dt.$$

This corresponds to the *Principle of Virtual Power* in analytical mechanics, cf. e.g. [73]. The tangent vector u is sometimes called *test function*. In the case that the pairing is nondegenerate, for example, in the finite dimensional case, we can write (2.1.15) as

$$\frac{d}{dt} D_2 L(x, v) = D_1 L(x, v) + \alpha, \tag{2.1.16}$$

which is the classical form of the Euler-Lagrange equation. Here the operators D_1, D_2 denote the partial derivatives of L with respect to configuration and velocity variables respectively.

2.2. On the Special Orthogonal Group $SO(3)$

Now we apply the Lagrange-d'Alembert Principle 2.1.9 to problems with the Special Orthogonal Group $SO(3)$ as configuration space and find the corresponding equations of motion. Recall that each element A in $SO(3)$ is an element in $GL(3)$, the group of all 3×3 nonsingular matrices, which satisfies the condition $A^T A = I$

and $\det(A) = 1$. Due to the constraints, the classical Euler-Lagrange Equation in local coordinates is not directly applicable. For the modeling problems regarding coupled rigid bodies, see e.g. [76].

Let the operator $\hat{}$ denote the natural isomorphism from $\mathbb{R}^3$ to $so(3)$, the space of 3×3 skew-symmetric matrices, defined by

$$\widehat{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}. \tag{2.2.1}$$

The following identities satisfied by this operator can be checked by direct computation. With $a, b, c \in \mathbb{R}^3$,

$$\hat{a}\, b = a \times b. \tag{2.2.2a}$$

$$[\hat{a}, \hat{b}] = \hat{a}\,\hat{b} - \hat{b}\,\hat{a} = \widehat{a \times b}. \tag{2.2.2b}$$

$$A\,\hat{a}\,A^T = \widehat{Aa}, \quad \text{for } A \in SO(3). \tag{2.2.2c}$$

$$tr(\hat{a}\,\hat{b}\,\hat{c}) = -(a \times b) \cdot c. \tag{2.2.2d}$$

$$tr(\hat{a}\,M\,\hat{a}^T) = a \cdot M^\circ\, a, \tag{2.2.2e}$$

where, for

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix},$$

we have

$$M^\circ = \begin{pmatrix} M_{22} + M_{33} & -M_{12} & -M_{13} \\ -M_{21} & M_{11} + M_{33} & -M_{23} \\ -M_{31} & -M_{32} & M_{11} + M_{22} \end{pmatrix}.$$

Moreover, in order to have

$$tr(\hat{a}\,M\,\hat{b}^T) = a \cdot M^\circ\, b, \tag{2.2.2f}$$

the matrix M needs to be symmetric. Here $tr(\cdot)$ denotes the trace of a matrix. These formula will be used later in calculations related to skew symmetric matrices.

Given $A \in SO(3)$, recall that $(A, A\hat{\Omega})$ characterizes elements in $TSO(3)$. In mechanics, the variable Ω corresponds to the instantaneous angular velocity of the

18

motion in body coordinates. To get a representation of the elements in the second tangent bundle $TTSO(3)$, we make the following observation. See [7] for relevant discussions.

$$\frac{d}{dt}A\hat{\Omega} = \dot{A}\hat{\Omega} + A\dot{\hat{\Omega}} = A(\hat{\Omega}\hat{\Omega} + \dot{\hat{\Omega}}).$$

Thus the *special* second tangent vectors must be of the form

$$\left(A, \ A\hat{\Omega}, \ A\hat{\Omega}, \ A(\hat{\Omega}\hat{\Omega} + \dot{\hat{\Omega}})\right).$$

In general, let $(A\hat{u}, W) \in T_{(A, \ A\hat{\Omega})}TSO(3)$. In order to have this vector generate a curve

$$\left(Ae^{\epsilon \hat{u}}, Ae^{\epsilon \hat{u}}(\hat{\Omega} + \epsilon \hat{w})\right) \in TSO(3)$$

which passes through $(A, \ A\hat{\Omega})$ when $\epsilon = 0$, we must have

$$W = \left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} Ae^{\epsilon \hat{u}}(\hat{\Omega} + \epsilon \hat{w}) = A(\hat{u}\hat{\Omega} + \hat{w}).$$

Consequently, any element in $TTSO(3)$ can be written as

$$\left(A, \ A\hat{\Omega}, \ A\hat{u}, \ A(\hat{u}\hat{\Omega} + \hat{w})\right). \tag{2.2.3}$$

Next we look for a canonical representation for an element in $T^*TSO(3)$, the dual space to the second tangent bundle. First, recall that the *trace* pairing in $GL(n)$

$$\langle A, B \rangle = \frac{1}{2}tr(A^T B). \tag{2.2.4}$$

This provides us with a standard way to define elements in $T^*SO(3)$, i.e. we could let $A\hat{a} \in T_A^*SO(3)$, and

$$\langle A\hat{a}, A\hat{u} \rangle = \frac{1}{2}tr(\hat{a}^T A^T A\hat{u}) = a \cdot u, \tag{2.2.5}$$

where $a \cdot u$ denotes the Euclidean inner product. This is the negative of the *Killing form* on the Lie Group $SO(3)$ [35]. Let $\omega(A, A\hat{\Omega}) = (\alpha, \beta) \in T_{(A, A\hat{\Omega})}^*TSO(3)$. We have

$$\omega(A, A\hat{\Omega})\left(A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})\right) = \frac{1}{2}tr(\alpha^T A\hat{u}) + \frac{1}{2}tr(\beta^T A(\hat{u}\hat{\Omega} + \hat{w})).$$

In order to have $a, b \in \mathbb{R}^3$, such that

$$\omega(A, A\hat{\Omega})\left(A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})\right) = a \cdot u + b \cdot w,$$

19

we ask

$$\alpha = A(\hat{b}\hat{\Omega} + \hat{a})$$

$$\beta = A\hat{b}.$$

Thus we obtain the representation for elements in $T^*TSO(3)$ as follows,

$$\left(A, \; A\hat{\Omega}, \; A(\hat{b}\hat{\Omega} + \hat{a}), \; A\hat{b} \right).$$ (2.2.6)

The above discussions are based on the representation $(A, A\hat{\Omega})$ for elements in $TSO(3)$, where Ω is in *body* coordinate. We could work out similar formulations bases on the representation $(A, \hat{\omega}A)$, where ω is the instantaneous angular velocity in *spatial* coordinate. We have the relationship $\omega = A\Omega$.

We remark here that these are the parametrizations of $TTSO(3)$ and $T^*TSO(3)$ which are *globally defined* via the embedding of $SO(3)$ in $GL(3)$. Our goal has been to make the pairing analogous to the Euclidean space. These global representations (2.2.3), (2.2.6) of the second tangent bundle and the dual of the second tangent bundle on $SO(3)$ also prove to be useful in finding the derivatives or variations of a function (Lagrangian) on $TSO(3)$ and in deriving the reduced Poisson bracket. These issues will be discussed in the following chapters in greater detail. In the following, we state the Lagrange-d'Alembert Principle in terms of these representations. For simplicity, we restrict ourselves for the moment to problems with configuration space $SO(3)$. We now show how to derive the dynamical equations of motion on that space.

Following an argument similar to the one used in deriving the classical Euler-Lagrange equation in local coordinates as in Section 2.1, we notice that all the definitions and identities corresponding to the local coordinates also hold here. We have the following theorem.

THEOREM 2.2.1

On $TSO(3)$, let a system be described by a Lagrangian L. The Lagrange-d'Alembert Principle in the invariant form (2.1.14) applied to the motions on $SO(3)$ gives us the Euler-Lagrange equation,

$$\langle \frac{d}{dt} D_2 L(A, \; A\hat{\Omega}), \; A\hat{u}\rangle = \langle D_1 L(A, \; A\hat{\Omega}), \; A\hat{u}\rangle + \langle \alpha, \; A\hat{u}\rangle,$$ (2.2.7)

$$\forall \; A\hat{u} \in T_A SO(3).$$

20

where α is the exterior force. (Here D_1, D_2 are the usual partial Fréchet differentials.)

Proof

Based on an argument similar to the one in Section 2.1, we have the following formulae. The key observation here is that with the representations (2.2.3) and (2.2.6), the space of vertical tangent vectors is isomorphic to the fibers in $TSO(3)$. In particular, vertical vector fields could be written as

$$(0, A\hat{w}), \quad A\hat{w} \in T_A SO(3),$$

and the horizontal 1-forms as

$$(A\hat{a}, 0), \quad A\hat{a} \in T_A^* SO(3).$$

The corresponding mappings γ, τ, τ_* are well defined as, cf. (2.1.1), (2.1.2), (2.1.3),

$$\gamma_{(A, \ A\hat{\Omega})} \cdot A\hat{u} \ = \ (0, A\hat{u}),$$

$$\tau_{(A, A\hat{\Omega})}(A(\hat{b}\hat{\Omega} + \hat{a}), A\hat{b}) \ = \ (A\hat{b}, 0),$$

$$\tau_{*(A, A\hat{\Omega})}(A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})) \ = \ (0, A\hat{u}).$$

A special vector field takes the form

$$X(A, \ A\hat{\Omega}) \ = \ (A\hat{\Omega}, A(\hat{\Omega}\hat{\Omega} + \hat{w})).$$

The differential form of the Lagrangian on $TSO(3)$ can be written as

$$dL(A, \ A\hat{\Omega}) \ (A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})$$

$$= \ \left[L(\ Ae^{t\hat{u}}, Ae^{t\hat{u}}(\hat{\Omega} + t\hat{w}) \right]$$

$$= \ \langle \ (D_1 L(A, \ A\hat{\Omega}), \ D_2 L(A, \ A\hat{\Omega})), \ (A\hat{u}, \ A(\hat{u}\hat{\Omega} + \hat{w})) \ \rangle.$$

The 1-form defined in (2.1.5) is now

$$\Theta_L \ = \ \tau \cdot dL \ = \ (D_2 L(A, \ A\hat{\Omega}), \ 0).$$

With proper interpretations of the pairing, the derivations in Sec. 2.1 give us the form of the Euler-Lagrange's equation (2.2.7) for a motion on $SO(3)$.

∎

21

Now we ascribe meanings to the *proper interpretations* via the following example.

EXAMPLE 2.2.2

Consider the motion of a free rigid body in space. The configuration space is $SO(3)$. Let $(A, A\hat{\Omega})$ be an element in $TSO(3)$ with the physical interpretation that A represents the attitude of the body and Ω is the instantaneous angular velocity in body coordinates. The Lagrangian of the system can be written as

$$L(A, A\hat{\Omega}) = \frac{1}{2}\Omega \cdot I\Omega, \tag{2.2.8}$$

where I is the moment of inertia of the rigid body. Now we find the differential of L in the form of (2.2.6). Let $(U, W) \in T_{(A, A\hat{\Omega})}TSO(3)$, which could be written as, cf. (2.2.3),

$$(A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})),$$

where $u, w \in \mathbb{R}^3$. It generates a curve in $TSO(3)$ given by

$$\left(Ae^{\epsilon\hat{u}}, Ae^{\epsilon\hat{u}}(\hat{\Omega} + \epsilon\hat{w})\right).$$

Hence we have, from (2.2.8),

$$dL(A, A\hat{\Omega})(U, W)$$
$$= \frac{d}{d\epsilon}\Bigg|_{\epsilon=0} L\left(Ae^{\epsilon\hat{u}}, Ae^{\epsilon\hat{u}}(\hat{\Omega} + \epsilon\hat{w})\right)$$
$$= \frac{d}{d\epsilon}\Bigg|_{\epsilon=0} (\Omega + \epsilon w)^T I(\Omega + \epsilon w)$$
$$= w \cdot I\Omega.$$

The canonical form for $dL(A, A\hat{\Omega})$ is $\left(A(\hat{b}\hat{\Omega} + \hat{a}), A\hat{b}\right)$, cf. (2.2.6), and

$$dL(A, A\hat{\Omega})(U, W) = a \cdot u + b \cdot w.$$

Thus $b = I\Omega$ and we obtain

$$D_1 L(A, A\hat{\Omega}) = A\widehat{\widehat{I\Omega}\hat{\Omega}},$$
$$D_2 L(A, A\hat{\Omega}) = A\widehat{I\Omega}.$$

By taking the time derivative of $D_2 L$, we get

$$\frac{d}{dt} D_2 L(A, A\hat{\Omega}) = \dot{A}\widehat{I\Omega} + A\widehat{I\dot{\Omega}} = A(\hat{\Omega}\widehat{I\Omega} + \widehat{I\dot{\Omega}}).$$

From Theorem 2.2.1,

$$\langle A(\hat{\Omega}\widehat{I\Omega} + \widehat{I\dot{\Omega}}), A\hat{u}\rangle = \langle A\widehat{I\Omega}\hat{\Omega}, A\hat{u}\rangle, \quad \forall A\hat{u} \in T_A SO(3),$$

or

$$\langle A(\widehat{I\dot{\Omega}} + \hat{\Omega}\widehat{I\Omega} - \widehat{I\Omega}\hat{\Omega}), A\hat{u}\rangle = 0, \quad \forall A\hat{u} \in T_A SO(3),$$

which implies

$$\langle A(I\dot{\Omega} + \Omega \times I\Omega)\hat{\ }, A\hat{u}\rangle = 0, \quad \forall A\hat{u} \in T_A SO(3),$$

or

$$(I\dot{\Omega} + \Omega \times I\Omega) \cdot u = 0, \quad \forall u \in \mathbb{R}^3.$$

Since the Euclidean inner product is nondegenerate, we conclude,

$$I\dot{\Omega} = -\Omega \times I\Omega, \tag{2.2.9}$$

which is exactly the Euler's equation for rigid body. Accompanied with the attitude equation,

$$\dot{A} = A\hat{\Omega},$$

the dynamical equations of a free rigid body are obtained.

∎

In the following chapters, we will use the procedure of this section as a model for working out the dynamics of more complicated problems. We remark that, under this framework, we may mix the local coordinate on some manifold and the parametrizations for $SO(3)$ in deriving the equations of motion. This is very helpful when we are dealing with systems which have a mixed configuration space (e.g. Cartesian products of a Lie group and a smooth manifold).

CHAPTER III

Gyroscopic Systems with Symmetry

In this chapter, we discuss a general framework for gyroscopic systems with symmetry, which include the simple mechanical systems with symmetry in the sense of Smale [64] as a special case. A variational principle which characterizes the relative equilibria is derived and applied to a particular example, namely, a mechanical system consisting of two rigid bodies connected by a ball-in-socket joint.

3.1. Preliminaries

In this section, we introduce a few notions in the theory of riemannian manifolds, Lie groups, and reduction which will be used frequently in subsequent discussions in this dissertation. Let $(Q, \ll \cdot, \cdot \gg)$ be a riemannian manifold with the riemannian metric $\ll \cdot, \cdot \gg$. We sometimes write

$$K(x)(v_x, \ w_x) \ = \ \ll v_x, \ w_x \gg_x, \tag{3.1.1}$$

for $x \in Q$, and $v_x, w_x \in T_xQ$. This Riemannian metric induces a vector bundle isomorphism

$$K^\flat \ : \ TQ \ \to \ T^*Q,$$

defined by

$$\langle K^\flat(v_x), \ w_x \rangle_x \ = \ \ll v_x, \ w_x \gg_x, \quad \text{for all } v_x, \ w_x \in T_xQ, \tag{3.1.2}$$

where $\langle \cdot, \cdot \rangle_x$ denotes the pairing between elements in T_x^*Q and T_xQ. By the Riesz Representation Theorem, this isomorphism is well defined and we may write

$$K^\sharp \ = \ (K^\flat)^{-1} \ : \ T^*Q \ \to \ TQ,$$

24

which is also a fiber-preserving mapping. We have the following relation. For $\alpha_x \in T^*Q$,

$$\langle \alpha_x, w_x \rangle_x = \ll K^\sharp \cdot \alpha_x, w_x \gg_x, \quad \text{for } w_x \in TQ. \tag{3.1.3}$$

The mappings $K^\flat$, $K^\sharp$ are both linear in their argument. Using the isomorphism $K^\sharp$, we define an inner product on T^*Q as

$$< \alpha_x, \beta_x >_{T^*Q} \stackrel{\triangle}{=} \ll K^\sharp \cdot \alpha_x, K^\sharp \cdot \beta_x \gg_x, \tag{3.1.4}$$

for $\alpha_x, \beta_x \in T_x^*Q$. Let G be a Lie group, $\Phi : G \times Q \rightarrow Q$ be a group action of G on the manifold Q. We shall use the notations $\Phi(g, x) \equiv \Phi_g(x) \equiv g \cdot x$ interchangeably to denote this action. We define the associated actions on TQ, T^*Q in the following way. The *tangent lift* Φ^T associated with Φ is defined as $\Phi_g^T \stackrel{\triangle}{=} T\Phi_g : TQ \rightarrow TQ$, or, in local coordinates,

$$\Phi_g^T(x, v) = (\Phi_g(x), T_x\Phi_g \cdot v).$$

The *cotangent lift* Φ^{T^*} associated with Φ on the cotangent bundle T^*Q is defined as

$$\Phi^{T^*} : G \times T^*Q \rightarrow T^*Q,$$

$$\Phi_g^{T^*}(\alpha_x) \stackrel{\triangle}{=} T^*\Phi_{g^{-1}} \cdot \alpha_x,$$

where $T^*\Phi_{g^{-1}}$ is the dual of $T\Phi_{g^{-1}}$. In local coordinates, we have

$$\langle \Phi_g^{T^*}(x, \alpha), (g \cdot x, v) \rangle_{g \cdot x} = \langle \alpha, T_{g \cdot x}\Phi_{g^{-1}} \cdot v \rangle_x. \tag{3.1.5}$$

The cotangent lift just defined is exactly the same notion as the *lifted action* in [2]. The former terminology is adopted here to indicate the space it acts on. It is easy to show that the tangent lift and cotangent lift are both well-defined actions on the spaces TQ and T^*Q respectively.

Now we review a few notions in the theory of Lie groups and reduction, cf. [2]. Let the Lie algebra of a Lie group G be denoted by $\mathcal{G}$, with its dual $\mathcal{G}^*$. Recall that the Lie algebra $\mathcal{G}$ is identified as the tangent space to G at the identity element e or, equivalently, the set of left invariant vector fields on G, cf. also [49]. Given $\xi \in \mathcal{G}$, for a group action on a manifold Q, we define

$$\xi_Q(x) \stackrel{\triangle}{=} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi_{\exp \epsilon \xi}(x) \quad \in T_xQ, \tag{3.1.6}$$

25

the *infinitesimal generator* of the action corresponding to ξ. Group G acts on $\mathcal{G}$ through the adjoint action

$$Ad \; : \; G \; \times \; \mathcal{G} \; \rightarrow \; \mathcal{G}$$
$$(\, g, \, \xi \,) \; \mapsto \; T_e(R_{g^{-1}} \circ L_g)\xi \; = \; Ad_g\xi, \tag{3.1.7}$$

where L_g, R_g denote the left and right translation of a group element by $g \in G$, respectively. The map $g \mapsto Ad_g$ is also called the *adjoint representation* of G in $\mathcal{G}$. The infinitesimal generator of this adjoint action

$$\xi_{\mathcal{G}}(\eta) \; = \; \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} Ad_{\exp \epsilon \xi}(\eta)$$

can be shown to be equal to the Lie bracket of ξ and η, namely,

$$\xi_{\mathcal{G}}(\eta) \; = \; [\, \xi, \, \eta \,] \; \overset{\triangle}{=} \; ad_\xi\eta. \tag{3.1.8}$$

The group G also acts on the dual of the Lie algebra $\mathcal{G}^*$ through the coadjoint action

$$Ad^* \; : \; G \; \times \; \mathcal{G}^* \; \rightarrow \; \mathcal{G}^*,$$
$$(\, g, \, \mu \,) \; \mapsto \; Ad^*_{g^{-1}}\mu,$$

which is defined by,

$$\langle\, Ad^*_g\mu, \, \xi \,\rangle \; \overset{\triangle}{=} \; \langle\mu, \, Ad_g\xi\rangle, \quad \forall\, \xi \in \mathcal{G}, \tag{3.1.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $\mathcal{G}^* \times \mathcal{G}$. The corresponding infinitesimal generator, $\xi_{\mathcal{G}^*}$ can be shown to be

$$\xi_{\mathcal{G}^*}(\mu) \; = \; -\langle\mu, \, [\xi, \, \eta]\rangle \; \overset{\triangle}{=} \; -\langle ad^*_\xi\mu, \, \eta\rangle, \tag{3.1.10}$$

for all $\eta \in \mathcal{G}$. We have the identity,

$$\langle ad^*_\xi\mu, \, \eta\rangle \; = \; \langle\mu, \, ad_\xi\eta\rangle. \tag{3.1.11}$$

Here, since $\mathcal{G}$ and $\mathcal{G}^*$ are both vector spaces, their tangent spaces are isomorphic to themselves. With the above construction, group G acts on Q and $\mathcal{G}^*$ through the actions Φ and Ad^* respectively. A map $J : Q \rightarrow \mathcal{G}^*$ is called Ad^*-*equivariant* if

$$J \circ \Phi_g \; = \; Ad^*_{g^{-1}} \circ J.$$

26

For $\mu \in \mathcal{G}^*$, we define the *isotropy subgroup* associated with it by

$$G_\mu = \{\, g \in G \; : \; Ad_g^*\mu = \mu \,\}, \tag{3.1.12}$$

with its Lie algebra

$$\mathcal{G}_\mu = \{\, \eta \in \mathcal{G} \; : \; ad_\eta^*\mu = 0 \,\}, \tag{3.1.13}$$

which is a subalgebra of $\mathcal{G}$.

Next we introduce the notion of invariance. We say that a riemannian metric is *G-invariant* if it is invariant under the pull back of the mapping Φ_g, i.e. for all $g \in G$, $\Phi_g^* \cdot K = K$, or in local coordinates,

$$K(x)(v,\ w) = K(g \cdot x)(T_x\Phi_g \cdot v,\ T_x\Phi_g \cdot w), \quad \forall\, g \in G,\ v,\ w \in T_xQ. \tag{3.1.14}$$

It follows that the inner product on T^*Q defined in (3.1.4) is invariant under the cotangent lift, namely,

$$< \alpha_x,\ \beta_x >_{T^*Q} \ = \ < \Phi_g^{T^*}\alpha_x,\ \Phi_g^{T^*}\beta_x >_{T^*Q}, \tag{3.1.15}$$

for all $g \in G$. This can be shown by using the identities,

$$K^\sharp \cdot \Phi_g^{T^*} \cdot \alpha_x = T_x\Phi_g \cdot K^\sharp \cdot \alpha_x, \quad \text{for } \alpha_x \in T^*Q, \tag{3.1.16a}$$

$$K^\flat \cdot T_x\Phi_g \cdot w_x = \Phi_g^{T^*} \cdot K^\flat \cdot w_x, \quad \text{for } w_x \in TQ, \tag{3.1.16b}$$

which follow easily from (3.1.2), (3.1.3), and (3.1.14). A smooth function $V : Q \to \mathbb{R}$ is a *G-invariant function* on the manifold if, for all $g \in G$,

$$V(\Phi_g(x)) = V(x). \tag{3.1.17}$$

Let Y be a vector field on Q. We say that Y is a *G-invariant vector field* if for all $g \in G$, $(\Phi_g)_* Y = Y$, or

$$Y(x) = T\Phi_g \cdot Y(g^{-1} \cdot x), \quad \text{for } x \in Q,\ g \in G. \tag{3.1.18}$$

Recall that a differential operator on the full tensor algebra can be defined from its restrictions on functions and vector fields through the Willmore Theorem [74].

Accordingly, the Lie derivative of a vector field on the tensor algebra can be found from its Lie derivative on functions (directional derivative) and Lie derivative on vector fields (Lie bracket). The following lemmas are essential to the developments in Chapter 5.

LEMMA 3.1.1

For $\eta \in \mathcal{G}$, the Lie derivatives

$$L_{\eta_Q} Y = 0, \tag{3.1.19}$$

$$L_{\eta_Q} K = 0. \tag{3.1.20}$$

Proof

Both Y and K are tensors on Q. For a general tensor t on Q, we have the following identity, cf. p. 90 in [2]. for a vector field $X \in \mathfrak{X}(Q)$,

$$L_X \mathrm{t}(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left((F_\epsilon^X)^* \mathrm{t}(x) - \mathrm{t}(x) \right), \tag{3.1.21}$$

where F_ϵ^X is the flow on Q associated with X. By choosing $X = \eta_Q$, the infinitesimal generator corresponding to η, we have

$$(F_\epsilon^X)^* Y = (\Phi_{\exp -\epsilon\eta})_* Y, \qquad (F_\epsilon^X)^* K = (\Phi_{\exp \epsilon\eta})^* K.$$

Th lemma follows then from (3.1.14), (3.1.18), and (3.1.21).

∎

Moreover, for two vector fields X_1, $X_2 \in \mathfrak{X}(Q)$, we have the following important property.

LEMMA 3.1.2

For $\eta \in \mathcal{G}$, we have

$$L_{\eta_Q} \ll X_1,\, X_2 \gg_x = \ll L_{\eta_Q} X_1,\, X_2 \gg_x + \ll X_1,\, L_{\eta_Q} X_2 \gg_x . \tag{3.1.22}$$

Proof

Here, $\ll X_1,\, X_2 \gg_x$ should be thought as a function on the manifold Q. The LHS in (3.1.22) is thus the Lie derivative of a function with respect to a vector field. Recall that a differential operator D on the full tensor algebra $\mathcal{T}(Q)$ has the following

property, cf. [2], p. 88. Letting $t \in \mathcal{T}_s^r(Q)$, $\alpha_1, \cdots, \alpha_r \in \varpi^1(Q)$, $X_1, \cdots, X_s \in \mathfrak{X}(Q)$, we have

$$
D\big(t(\alpha_1, \cdots, \alpha_r, X_1, \cdots, X_s)\big) = (Dt)(\alpha_1, \cdots, \alpha_r, X_1, \cdots, X_s)
$$
$$
+ \sum_{j=1}^{r} t(\alpha_1, \cdots, D\alpha_j, \cdots, \alpha_r, X_1, \cdots, X_s)
$$
$$
+ \sum_{j=1}^{r} t(\alpha_1, \cdots, \alpha_r, X_1, \cdots, DX_j, \cdots, X_s).
$$

$$(3.1.23)$$

The Lie derivative L_{η_Q} is a well-defined differential operator. By replacing D by L_{η_Q}, replacing t by K in (3.1.23), and using Lemma 3.1.1, the desired identity is obtained.

∎

3.2. System Description

With the notions introduced in the previous section, we are ready to define *gyroscopic systems with symmetry*.

DEFINITION 3.2.1

A *Gyroscopic System with Symmetry* is a 5-tuple, (Q, K, Y, V, G), where

(1) (Q, K) is a Riemannian manifold.

(2) Y is a vector field on Q, which is called a *gyroscopic field*.

(3) V is a function on Q, which is called a *potential*.

(4) G is a Lie group with an action on Q, which leaves K, Y, V invariant and is called the *symmetry group*.

(5) Within the framework of lagrangian mechanics (cf. Chapter 2), the system is characterized by a Lagrangian $L : TQ \to \mathbb{R}$ in the form of

$$
L(v_x) = \frac{1}{2} K(x)(v_x, \; v_x) + K(x)(v_x, \; Y(x)) - V(x). \qquad (3.2.1)
$$

∎

The name "gyroscopic" comes from the second term in the Lagrangian (3.2.1), which includes the gyroscopic field Y. This term is *linear* in the velocity variables and

is responsible for the paradoxical behavior of gyroscopes. The Coriolis force in a rotating reference system and the magnetic force due to electric currents are examples of the effect of gyroscopic terms in the Lagrangian function. To see how the gyroscopic term enters the dynamical equations, we restrict our attention for the moment to a gyroscopic system (without symmetry consideration) on $\mathbb{R}^n$ described by the Lagrangian

$$L(x,\, v) \; = \; \frac{1}{2} v^T M(x) v \; + \; v^T \tilde{Y}(x) \; - \; V(x), \tag{3.2.2}$$

where $M(x)$ is a symmetric positive-definite second-order tensor, vectors x, $v(=\dot{x})$ are in $\mathbb{R}^n$, $\tilde{Y}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and V is a real-valued function. This is a gyroscopic system in the sense of Definition 3.2.1 with

$$K(x)(v,\, v) \; = \; v^T M(x) v, \quad \text{and} \quad Y(x) \; = \; M(x)^{-1}\tilde{Y}(x).$$

Abstractly, $\tilde{Y}$ should be regarded as a 1-form in T^*Q.

To obtain the dynamical equations associated with the Lagrangian in (3.2.2), we invoke the classical Euler-Lagrange equations, cf. (2.1.16). First, we find

$$\frac{\partial L}{\partial v} \; = \; M(x) \cdot v \; + \; \tilde{Y}(x).$$

By taking time derivatives, we get

$$\frac{d}{dt}\frac{\partial L}{\partial v} \; = \; M(x) \cdot \dot{v} \; + \; \left(\frac{\partial M}{\partial x}(x) \cdot v \right) \cdot v \; + \; \frac{\partial \tilde{Y}}{\partial x}(x) \cdot v,$$

where $\partial M/\partial x$ is a third-order tensor, and $\partial \tilde{Y}/\partial x$ is a second-order tensor. Recall the definition of third-order tensors through triads,

$$(\mathbf{abc}) \cdot \mathbf{w} \; = \; \mathbf{ab}(\mathbf{c} \cdot \mathbf{w}),$$

$$(\mathbf{abc}) \; : \; (\mathbf{uv}) \; = \; \mathbf{a}(\mathbf{b} \cdot \mathbf{u})(\mathbf{c} \cdot \mathbf{v}),$$

where " $\cdot$ " between vectors denotes some scalar product on $\mathbb{R}^n$. Here, and in what follows, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ denote vectors in $\mathbb{R}^n$. We have the identity, for a third-order tensor $\mathbf{T}$,

$$\mathbf{T} \; : \; (\mathbf{uv}) \; = \; (\mathbf{T} \cdot \mathbf{v}) \cdot \mathbf{u}.$$

With these notations in Tensor Algebra, see, e.g. [4], we may write

$$\frac{d}{dt}\frac{\partial L}{\partial v} = M(x) \cdot \dot{v} + \frac{\partial M}{\partial x}(x) : vv + \frac{\partial \tilde{Y}}{\partial x}(x) \cdot v. \qquad (3.2.3)$$

Next we find the partial derivative of L with respect to x as follows. By definition,

$$\frac{\partial}{\partial x}(v^T M(x)v) \cdot \mathbf{w} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} v^T M(x + \epsilon\,\mathbf{w})v,$$

$$= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} M(x + \epsilon\,\mathbf{w}) : (vv) = \left(\frac{\partial M}{\partial x}(x) \cdot \mathbf{w}\right) : (vv).$$

We need the following lemma.

<u>LEMMA 3.2.2</u>

Let $\mathbf{T}$ be a third-order tensor, we have the identity

$$(\mathbf{T} \cdot \mathbf{w}) : (\mathbf{uv}) = (\mathbf{T}^* : (\mathbf{uv})) \cdot \mathbf{w}, \quad \forall\ \mathbf{u}, \mathbf{v}, \mathbf{w},$$

where $\mathbf{T}^*$ is a third-order tensor defined by

$$\mathbf{T}^* \cdot \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w} \triangleq \mathbf{T} \cdot \mathbf{w} \cdot \mathbf{u} \cdot \mathbf{v}, \quad \forall\ \mathbf{u}, \mathbf{v}, \mathbf{w},$$

and is called the *cyclic transpose* of $\mathbf{T}$.

Proof

Since the triads form a basis for the space of third-order tensors, we only need to prove the lemma for triads. By definition, we have

$$((\mathbf{abc}) \cdot \mathbf{w}) : (\mathbf{uv}) = (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v})(\mathbf{c} \cdot \mathbf{w}).$$

It is easy to check that for a triad

$$(\mathbf{abc})^* = \mathbf{cab}.$$

Since we have,

$$((\mathbf{cab}) : (\mathbf{uv})) \cdot \mathbf{w} = (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v})(\mathbf{c} \cdot \mathbf{w}),$$

the lemma follows.

∎

From this lemma, we can write

$$\left(\frac{\partial M}{\partial x}(x) \cdot \mathbf{w}\right) \; : \; (vv) \; = \; \left(\frac{\partial M^*}{\partial x}(x) : (vv)\right) \cdot \mathbf{w}.$$

Thus we get

$$\frac{\partial}{\partial x}(v^T M(x)v) \; = \; \frac{\partial M^*}{\partial x}(x) : (vv). \tag{3.2.4}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial x}(v^T \tilde{Y}(x)) \cdot \mathbf{w} &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} v^T \tilde{Y}(x + \epsilon \, \mathbf{w}) \\
&= \left(\frac{\partial \tilde{Y}}{\partial x}(x) \cdot \mathbf{w}\right) \cdot v \\
&= \left(\frac{\partial \tilde{Y}^T}{\partial x}(x) \cdot v\right) \cdot \mathbf{w}
\end{aligned}$$

where the superscript ^{T} denotes the transpose of the second order tensor. Thus we have

$$\frac{\partial}{\partial x}(v^T \tilde{Y}(x)) \; = \; \frac{\partial \tilde{Y}^T}{\partial x}(x) \cdot v. \tag{3.2.5}$$

By substituting (3.2.3),(3.2.4),(3.2.5) in the Euler-Lagrange equations, we get

$$M(x) \cdot \dot{v} + \frac{\partial M}{\partial x}(x) : (vv) + \frac{\partial \tilde{Y}}{\partial x}(x) \cdot v = \frac{1}{2}\frac{\partial M^*}{\partial x}(x) : (vv) + \frac{\partial \tilde{Y}^T}{\partial x}(x) \cdot v - \frac{\partial V}{\partial x}(x).$$

By combining terms, it follows that,

$$M(x) \cdot \dot{v} = \left(\frac{1}{2}\frac{\partial M^*}{\partial x}(x) - \frac{\partial M}{\partial x}(x)\right) : (vv) - \left(\frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}^T}{\partial x}(x)\right) \cdot v - \frac{\partial V}{\partial x}(x).$$

Define

$$\mathcal{R} \; \triangleq \; \frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}^T}{\partial x}(x), \tag{3.2.6a}$$

$$\mathcal{T} \; \triangleq \; \frac{1}{2}\frac{\partial M^*}{\partial x}(x) - \frac{\partial M}{\partial x}(x). \tag{3.2.6b}$$

The equations of motion can be then written as, by noting that $v = \dot{x}$,

$$M(x) \cdot \ddot{x} \; = \; \mathcal{T} \cdot \dot{x} \cdot \dot{x} \; - \; \mathcal{R} \cdot \dot{x} \; - \; \frac{\partial V}{\partial x}(x). \tag{3.2.7}$$

Note that $\mathcal{R}$ is a skew-symmetric tensor, thus the second term in the RHS gives the gyroscopic force in the dynamical equations as discussed in [13]. We remark that the

component form of $M(x)^{-1}T$ is nothing but the Christoffel symbol associated with the geodesic flow. Cf. e.g. [2].

EXAMPLE 3.2.3

We consider the dynamical system treated in [13] in the following form.

$$\ddot{x} = -\alpha x - g\dot{y},$$
$$\ddot{y} = -\beta y + g\dot{x}. \tag{3.2.8}$$

The skew terms in velocities $g\dot{y}$ and $g\dot{x}$ constitute the *gyroscopic forces* which do no net work but affect the stability of the system. It is easily checked that this is a gyroscopic system with the Lagrangian in the form of (3.2.2) with the following entities,

$$M(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Y}(x,y) = \begin{pmatrix} gy \\ 0 \end{pmatrix}, \quad V(x,y) = \frac{1}{2}(x\ y) \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

∎

Now we go back to the abstract framework of the gyroscopic systems with symmetry. We remark first that a simple mechanical system with symmetry in the sense of Smale [64] is a special case of the gyroscopic systems with symmetry. We simply take $Y = 0$ and consider the quadruple (Q, K, V, G). In this dissertation we show that many key results in the category of simple mechanical systems with symmetry can be extended to gyroscopic systems. For a gyroscopic system with symmetry with the Lagrangian (3.2.1), we have the following lemma.

LEMMA 3.2.4

L is invariant under the tangent lift Φ^T.

Proof

We need to show that, for all $g \in G$, $(\Phi_g^T)^* L = L$, or,

$$L(\Phi_g^T \cdot v_x) = L(v_x).$$

We proceed as follows.

$$L(\Phi_g^T \cdot v_x) = \frac{1}{2} K(g \cdot x)(T_x \Phi_g \cdot v_x, T_x \Phi_g \cdot v_x)$$
$$+ K(g \cdot x)(T_x \Phi_g \cdot v_x, Y(g \cdot x)) - V(g \cdot x).$$

33

By the G-invariance of the gyroscopic field Y, (3.1.18), we can write

$$L(\Phi_g^T \cdot v_x) = \frac{1}{2} K(g \cdot x)(T_x \Phi_g \cdot v_x, \; T_x \Phi_g \cdot v_x)$$

$$+ \; K(g \cdot x)(T_x \Phi_g \cdot v_x, \; T_x \Phi_g \cdot Y(x)) \; - \; V(g \cdot x).$$

$$= \frac{1}{2} K(x)(v_x, \; v_x) \; + \; K(x)(v_x, \; Y(x)) \; - \; V(x)$$

$$= L(v_x),$$

by the invariance property of the metric K and the potential V.

■

The Legendre transformation corresponding to this Lagrangian can be found as follows, cf. (2.1.7),

$$\langle \ell_L(v_x), \; w_x \rangle_x = D_2 L(v_x) \cdot w_x \; = \; \left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} L(v_x + \epsilon w_x),$$

$$= \left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} \ll v_x + \epsilon w_x, \; v_x + \epsilon w_x \gg_x \; + \; \ll v_x + \epsilon w_x, \; Y(x) \gg_x \; - V(x),$$

$$= \; \ll v_x + Y(x), \; w_x \gg_x,$$

$$= \; \langle K^\flat(v_x + Y(x)), \; w_x \rangle_x.$$

We thus have

$$\ell_L(v_x) \; = \; K^\flat(v_x + Y(x)). \tag{3.2.9}$$

LEMMA 3.2.5

The Lagrangian L in the form of (3.2.1) is hyperregular.

Proof

The inverse of the Legendre transformation can be found as, for $\alpha_x \in T^*Q$,

$$\ell_L^{-1}(\alpha_x) \; = \; K^\sharp(\alpha_x) \; - \; Y(x). \tag{3.2.10}$$

It follows that ℓ_L is a diffeomorphism, and thus L is hyperregular.

■

From Lemma 3.2.5, and the comment after Lemma 2.1.2, the space $(TQ, \; \Omega_L = -d\Theta_L)$ is a symplectic manifold, where the symplectic form Ω_L is defined as in (2.1.6) through the 1-form Θ_L, which in turn can be written as

$$\Theta_L(v_x) \cdot (u, w) = K(x)\big(v + Y(x), \, u\big).$$

With Lemma 3.2.4, the group G acts on TQ through the tangent lift Φ^T as a symmetry group. It can be further verified that this action is symplectic, namely, $(\Phi^T)^* \Omega_L = \Omega_L$. Within this framework, the momentum mapping $J : TQ \to \mathcal{G}^*$ can be thus defined such that the infinitesimal generator of the action Φ^T corresponding to $\xi \in \mathcal{G}$ is the vector field induced by the function

$$\langle J, \, \xi \rangle \ : \ TQ \ \to \ \mathbb{R},$$

through the symplectic structure, cf. (2.1.12b). Consequently, we have the following theorem.

THEOREM 3.2.6

The gyroscopic system with symmetry (Q, K, Y, V, G) has the following properties.

(i) The 1-form corresponding to L defined in (2.1.5) is invariant under the tangent lift, i.e.

$$(\Phi_g^T)^* \, \Theta_L \ = \ \Theta_L.$$

(ii) There is an associated Ad^*-equivariant momentum mapping $J : TQ \to \mathcal{G}^*$, in the form of

$$J(v_x)(\xi) \ = \ \langle \ell_L(v_x), \, \xi_Q(x) \rangle_x \ = \ \ll v_x + Y(x), \, \xi_Q(x) \gg_x, \qquad (3.2.11)$$

where $\xi \in \mathcal{G}$ is an element in the Lie algebra of G, $\xi_Q(x)$ denotes the infinitesimal generator of ξ on Q. The notation $\mathcal{G}^*$ denotes the dual of the Lie algebra $\mathcal{G}$.

(iii) The momentum mapping defined in (3.2.11) is a vector-valued integral of any vector field induced by a G-invariant function on TQ through an analogous formula in (2.1.12). In particular, it is an integral of the Lagrangian vector field X_{H_L}.

Proof

For (i), we note that $(\Phi_g^T)^* L \ = \ L$, by Lemma 3.2.4. Since the exterior differentiation commute with the pull back operator, (i) follows immediately. Statements (ii), and (iii) can be shown by directly applying Theorem 4.2.2 and Corollary 4.2.14. in [2].

∎

35

The quadruple $(TQ, \Omega_L, \Phi^T, J)$ is an example of *hamiltonian G-space*. The energy function for the gyroscopic system can be found as, cf. (2.1.11),

$$
\begin{aligned}
H_L(v_x) &= \langle \ell_L(v_x), v_x \rangle_x - L(v_x), \\
&= \ll v_x + Y(x), v_x \gg_x - \frac{1}{2} \ll v_x, v_x \gg_x - \ll v_x, Y(x) \gg_x + V(x), \\
&= \frac{1}{2} \ll v_x, v_x \gg_x + V(x).
\end{aligned}
\tag{3.2.12}
$$

The energy function for the system is not affected by the gyroscopic field Y. However, the dynamics are different from what one would see if $Y = 0$. . The differences in the dynamical behavior inherit from the different symplectic 2-form Ω_L associated with different Lagrangian L. In particular, the gyroscopic term in the Lagrangian gives rise to the *magnetic terms* in the symplectic 2-form. On the other hand, on the momentum phase space T^*Q, the Hamiltonian associated with the system is

$$
\begin{aligned}
H(\alpha_x) &= H_L \circ \ell_L^{-1}(\alpha_x), \\
&= \frac{1}{2} \ll K^\sharp(\alpha_x) - Y(x), K^\sharp(\alpha_x) - Y(x) \gg_x + V(x), \\
&= \frac{1}{2} < \alpha_x - K^\flat(Y(x)), \alpha_x - K^\flat(Y(x)) >_{T^*Q} + V(x).
\end{aligned}
\tag{3.2.13}
$$

Consequently, on the momentum phase space, the Hamiltonian is affected by the gyroscopic term through the momentum shift, with the canonical 2-form ω_0 unchanged.

EXAMPLE 3.2.7

We consider again the system in Example 3.2.3. The energy associated with (3.2.15) on TQ is

$$
H_L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + \alpha x^2 + \beta y^2 \right),
$$

with the symplectic 2-form in matrix representation

$$
([\Omega_L]^{-1})^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -g \\ 0 & -1 & g & 0 \end{pmatrix}.
\tag{3.2.14}
$$

This can be checked from the equation, cf. (2.1.12c),

$$\dot{\mathbf{x}} = X_{H_L}(\mathbf{x}) = \left([\Omega_L]^{-1}\right)^T \nabla_{\mathbf{x}} H_L,$$

where $\mathbf{x} = (x, y, \dot{x}, \dot{y})$. The right-lower 2×2 block in $\left([\Omega_L]^{-1}\right)^T$ is called the magnetic part. On the other hand, on T^*Q, we have the conjugate momentum variables defined by

$$p_1 = \dot{x} + gy,$$

$$p_2 = \dot{y}.$$

The dynamical equation (3.2.8) can be written as

$$\dot{x} = p_1 - gy,$$

$$\dot{y} = p_2,$$

$$\dot{p}_1 = -\alpha x,$$

$$\dot{p}_2 = -\beta y + g(p_1 - gy),$$

which is a hamiltonian system with the Hamiltonian function

$$H(x, y, p_1, p_2) = \frac{1}{2}\left((p_1 - gy)^2 + p_2^2 + \alpha x^2 + \beta y^2\right).$$

The symplectic structure is the canonical symplectic 2-form ω_0, i.e. in matrix representation,

$$[\omega_0] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

∎

In summary, the gyroscopic term affects the symplectic 2-form on TQ side while, on T^*Q side, it affects the Hamiltonian function. To gain more insight about how the gyroscopic field enters the symplectic structure Ω_L, we consider an even simpler case than (3.2.2). We assume that the second-order tensor M is independent of x in (3.2.2). It can be easily found that the symplectic 2-form is now, cf. (2.1.6),

$$\Omega_L(q, v)\left((u_1, w_1), (u_2, w_2)\right)$$

$$= \frac{\partial \tilde{Y}}{\partial x} \cdot u_2 \cdot u_1 + M \cdot w_2 \cdot u_1 - \frac{\partial \tilde{Y}}{\partial x} \cdot u_1 \cdot u_2 - M \cdot w_1 \cdot u_2$$

$$= \begin{pmatrix} u_1 & w_1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \tilde{Y}}{\partial x} - \frac{\partial \tilde{Y}}{\partial x}^T & M \\ -M^T & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ w_2 \end{pmatrix}.$$

37

The block $\frac{\partial \tilde{Y}}{\partial x} - \frac{\partial \tilde{Y}}{\partial x}^T$ is the magnetic term.

3.3. Reductions

For the gyroscopic systems with symmetry (Q, K, Y, V, G), we may reduce the system dynamics to a lower order system by utilizing the symmetry properties. The reduction process has a long history. For Jacobi and Liouville[2] [5], this meant we could reduce the Hamilton's equation with some first integrals in involution. For Routh [58], this meant a process of eliminating ignorable variables. In the following, we shall discuss the reduction of our system from two points of view, namely, symplectic reduction and Poisson reduction.

First, we perform the symplectic reduction in the sense of [48]. As discussed in Chapter 2, (TQ, Ω_L) is a well-defined symplectic manifold, since L is hyperregular (see Lemma 3.2.5). By the Property (i) in Theorem 3.2.6, the Lie group G acts on Q symplectically, i.e. preserving the symplectic structure. Also, from Property (ii) in Theorem 3.2.6, there is an Ad^*-equivariant momentum mapping J for this action. Thus all the conditions in the Symplectic Reduction Theorem, see Theorems 4.3.1, 4.3.5, pp. 299, 304 in [2] are satisfied, we can thus state the following reduction theorem corresponding to the gyroscopic systems with symmetry.

THEOREM 3.3.1

Consider the gyroscopic system with symmetry (Q, K, Y, V, G). Assume that $\mu \in \mathcal{G}^*$ is a regular value of the momentum mapping J, as defined in (3.2.11), and that the isotropy subgroup G_μ, defined by

$$G_\mu = \{ g \in G : Ad^*_{g^{-1}}\mu = \mu \},$$

under the Ad^* action on $\mathcal{G}^*$ acts freely and properly on $J^{-1}(\mu)$, then

$$(TQ)_\mu \stackrel{\triangle}{=} J^{-1}(\mu) / G_\mu,$$

has a unique symplectic form Ω_μ with the property

$$\pi_\mu^* \Omega_\mu = i_\mu^* \Omega_L,$$

where $\pi_\mu : J^{-1}(\mu) \to (TQ)_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \hookrightarrow TQ$ is the inclusion map. Moreover, the flow F_t of X_{H_L} induces a flow F_t^μ on $(TQ)_\mu$ satisfying

$$\pi_\mu \cdot F_t = F_t^\mu \cdot \pi_\mu.$$

This flow is a Hamiltonian flow on $(TQ)_\mu$ with a Hamiltonian function H_L^μ satisfying

$$H_L^\mu \cdot \pi_\mu = H_L \cdot i_\mu,$$

with respect to the symplectic structure Ω_μ.

∎

The function H_L^μ on the reduced space is called the *reduced energy*. The corresponding vector field $X_{H_L^\mu}$ on the reduced space $(TQ)_\mu$ is called the *reduced vector field*. With the symplectic reduction, we thus first restrict our consideration to the level sets of the momentum mapping, and then factor out the isotropy group.

Next, we consider the Poisson reduction [47]. We first review basic framework for a Poisson manifold. A Poisson manifold P is a smooth manifold equipped with an $\mathbb{R}$-bilinear map (Poisson structure) on the space of smooth functions,

$$\{\cdot, \cdot\}_P : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$$

satisfying the axioms, for $f, g \in C^\infty(P)$,

(i) $\{f, g\}_P = -\{g, f\}_P$

(ii) $\{fg, h\}_P = g\{f, h\}_P + f\{g, h\}_P$

(iii) $\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0.$

Associated to a Poisson structure, there is a unique twice contravariant skew-symmetric, smooth tensor field Λ on P such that

$$\{f, g\}_P = \Lambda(df, dg),$$

where df, dg are differentials of f, g, respectively. The tensor field Λ defines a vector-bundle morphism,

$$\Lambda^{\#} : T^*P \rightarrow TP$$

$$\alpha_x \mapsto \Lambda^{\#}(\alpha_x) \in T_xP$$

satisfying,

$$\beta_x\left(\Lambda^{\#}(\alpha_x)\right) = \Lambda\left(x\right)(\alpha_x, \beta_x) \text{ for all } \beta_x \in T_x^*P.$$

Let G be a Lie group and let $\Psi : G \times P \rightarrow P$, $(g,x) \mapsto \Psi_g(x)$, be a group action such that $\Psi_g(\cdot)$ is a Poisson morphism for every $g \in G$, i.e. $\Psi_g : P \rightarrow P$ is an isomorphism and preserves the Poisson structure. Suppose that the action is proper and free. Then the quotient space P/G is a manifold which carries a Poisson structure $\{\cdot, \cdot\}_{P/G}$ induced from the one on P satisfying, for $f, g \in C^{\infty}(P/G)$,

$$\{f, g\}_{P/G} \circ \pi = \{f \circ \pi, g \circ \pi\}_P. \tag{3.3.1}$$

Here $\pi : P \rightarrow P/G$ is the canonical projection. By construction, it is a Poisson morphism.

G-equivariant dynamics on P induce dynamics on P/G. Suppose $h : P \rightarrow \mathbb{R}$ is a G-invariant Hamiltonian function on P, i.e.,

$$h(\Psi_g(x)) = h(x) \ \forall g \in G.$$

Define a vector field X_h through

$$X_h[f] = \{f, h\}_P \ \forall f \in C^{\infty}(P). \tag{3.3.2}$$

The Hamiltonian h descends to $\bar{h} : P/G \rightarrow \mathbb{R}$ and determines a *Poisson-reduced* dynamics $X_{\bar{h}}$ on P/G by

$$X_{\bar{h}}[\tilde{f}] = \{\tilde{f}, \bar{h}\}_{P/G} \ \forall \tilde{f} \in C^{\infty}(P/G). \tag{3.3.3}$$

Here $\bar{h}([x]) = h(x)$ for an equivalence class $[x]$ in P/G.

Recall that the symplectic manifold (TQ, Ω_L) has a canonical Poisson structure induced from the symplectic structure, namely, for $f, g \in C^{\infty}(TQ)$,

$$\{f, g\}_L(v_x) \triangleq df(v_x) \cdot X_g(v_x) \equiv \Omega_L(v_x)(\Pi_L(df), \Pi_L(dg)), \tag{3.3.4}$$

cf. (2.1.10). Since the energy function H_L is G-invariant, we carry out the Poisson reduction as follows. Assume G acts on TQ freely and properly. Let $\tilde{\tau}$ be the projection from TQ to TQ/G, $\tilde{f}, \tilde{g} \in C^\infty(TQ/G)$, the induced Poisson bracket of $\tilde{f}$ and $\tilde{g}$ is defined analogous to (3.3.1) as

$$\{\tilde{f}, \ \tilde{g}\}_i \circ \tilde{\tau} \ = \ \{\tilde{f} \circ \tilde{\tau}, \ \tilde{g} \circ \tilde{\tau}\}_L. \tag{3.3.5}$$

Within the framework of Poisson reduction, we have the following elements.

$$\tilde{H}_L \circ \tilde{\tau}(v_x) \ = \ H_L(v_x), \tag{3.3.6}$$

$$X_{\tilde{H}_L}[\tilde{f}] \ = \ \{\tilde{f}, \ \tilde{H}_L\}_i, \quad \forall \ \tilde{f} \in C^\infty(TQ/G). \tag{3.3.7}$$

Here the vector field $X_{\tilde{H}_L}$ is called the *projected Hamiltonian vector field* on TQ/G.

The reductions discussed here is on the Lagrangian side, or TQ side. We could perform similar reduction process on T^*Q side, or Hamiltonian side, by noting that the Hamiltonian function on T^*Q, namely H in (3.2.13), is invariant under the cotangent lift Φ^{T^*} (this could be derived from (3.1.15), (3.1.16)). This is usually the setting discussed in the literature regarding simple mechanical systems with symmetry. For comparison, we include the framework of gyroscopic systems with symmetry on the T^*Q side.

DEFINITION 3.3.2

A *Gyroscopic System with Symmetry* is a 5-tuple, (Q, K, Y, V, G), in which

(1) (Q, K) is a Riemannian manifold.

(2) Y is a vector field on Q, a *gyroscopic field*.

(3) V is a function on Q, a *potential*.

(4) G is a Lie group with an action on Q, which leaves K, Y, V invariant and is called the *symmetry group*.

(5) Within the framework of hamiltonian mechanics, the system is characterized by a Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$ in the form of

$$H(\alpha_x) \ = \ \frac{1}{2} < \alpha_x - K^\flat(Y(x)), \ \alpha_x - K^\flat(Y(x)) >_{T^*Q} + \ V(x). \tag{3.3.8}$$

where $< \cdot, \cdot >$ is the induced metric on T^*Q defined in (3.1.4).

$\blacksquare$

Since the two definitions 3.2.1 and 3.3.4 are equivalent, we will use the terminology *gyroscopic systems with symmetry* to refer either one of them, depending on what the underlying space is. The reduction on T^*Q side could be performed in a similar way. The symplectic manifold is (T^*Q, ω_0), cf. (2.1.8), with the corresponding momentum mapping,

$$J : T^*Q \rightarrow \mathcal{G}^*,$$

$$\langle J(\alpha_x), \xi \rangle = \langle \alpha_x, \xi_Q(x) \rangle_x.$$

(3.3.9)

These two reduction processes are equivalent, but the one on TQ side seems to be more intuitive.

3.4. Principle of Symmetric Criticality

In this section, we introduce the notion of relative equilibria and discuss their characterization. The concept of relative equilibrium goes back to Poincaré. With the symplectic reduction process, we define the notion of relative equilibrium as follows, cf. Theorem 3.3.1.

DEFINITION 3.4.1

A point v_x in TQ is called a *relative equilibrium* if $\pi_\mu(v_x) \in (TQ)_\mu$ is a fixed point for the reduced vector field $X_{H_L^\mu}$, where $\mu = J(v_x)$.

∎

Within the framework of Poisson reduction, we may define a similar notion, cf. (3.3.6), (3.3.7),

DEFINITION 3.4.2

A point v_x in TQ is called a *relative equilibrium* for X_{H_L} if

$$X_{\tilde{H}_L}(\tilde{\tau}(v_x)) = 0.$$

∎

It turns out that the two notions of relative equilibrium, Definition 3.4.1, 3.4.2, are equivalent. It can be shown, cf. [2], that, for both cases, v_x is a relative equilibrium *iff*

there exists a $\xi \in \mathcal{G}$ such that the flow of X_{H_L},

$$F^t_{X_{H_L}} (v_x) = \Phi_{exp(t\xi)} (v_x), \qquad (3.4.1)$$

i.e. the dynamical orbit is simply a group orbit. Thus if the observer were to be set in uniform motion according to the one-parameter group $exp(t\xi)$, then for such a moving observer, a relative equilibrium will appear to be stationary. For instance, if $G = SO(3)$, then $F^t_{X_{H_L}} (v_x)$ corresponds to a uniform rotation about a fixed axis ξ in space with the rotational speed $|\xi|$. In celestial mechanics, a relative equilibrium corresponds to exactly the state of circular motion of the bodies.

To characterize the relative equilibrium, we recall the following Souriau-Smale-Robbin Relative Equilibrium Theorem recast for our problem.

THEOREM 3.4.3

$v_x \in TQ$ is a relative equilibrium for X_H iff there exists a $\xi \in \mathcal{G}$ such that v_x is a critical point of

$$H_\xi \equiv H_L - \langle J, \xi \rangle, \qquad (3.4.2)$$

where $\langle J, \xi \rangle : TQ \to R$ is a real-valued function given by $v_x \mapsto \langle J(v_x), \xi \rangle$.

■

In particular, for the gyroscopic systems with symmetry, we have, cf. (3.3.9), (3.2.12),

$$
\begin{aligned}
H_\xi(v_x) &= \frac{1}{2} \ll v_x,\ v_x \gg_x + V(x) - \ll v_x + Y(x),\ \xi_Q(x) \gg_x, \\
&= \frac{1}{2} \ll v_x - \xi_Q(x),\ v_x - \xi_Q(x) \gg_x \\
&\quad + V(x) - \ll Y(x),\ \xi_Q(x) \gg_x -\frac{1}{2} \ll \xi_Q(x),\ \xi_Q(x) \gg_x.
\end{aligned}
\qquad (3.4.3)
$$

From Theorem 3.4.3, it is then easy to check that the necessary conditions for v_x to be a relative equilibrium are

$$v_x = \xi_Q(x), \qquad (3.4.4)$$

and

43

$$d_x[V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x] = 0.$$

We thus have the following algorithm (*principle of symmetric criticality*) to find relative equilibria.

ALGORITHM 3.4.4

    0. Pick $\xi \in \mathcal{G}$.

    1. Search for the critical points x_e of the function

$$V_\xi : Q \to R$$

$$V_\xi(x) \triangleq V(x) - \ll Y(x), \xi_Q(x) \gg_x - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x \tag{3.4.5}$$

    2. Substitute x_e in (3.4.4) to find the corresponding $v_e = \xi_Q(x_e)$.

                                                                         ■

We note that the computation in step 1 is fully on the configuration space. Thus the process of searching for a relative equilibrium is greatly simplified. We remark that, for simple mechanical systems with symmetry, the principle of symmetric criticality stated above appears as Theorem 1.1 in Part II of Smale[64]. Smale also notes that special versions have been known earlier, e.g. in the study of symmetric geodesics. See also p. 355 of [2], Theorem 16.7 in Hermann[31], Arnold[6], and Palais[51]. Here the *augmented potential* function V_ξ has one additional term to accommodate the gyroscopic effects. Through this term, we may change the number of critical points as well as the locations of them. Consequently, the phase portrait will be changed. This provides us an efficient tool to *control* the phase portrait. These comments will be made clear in the following chapters.

There is an additional symmetry in the augmented potential V_ξ. First, we define the *stabilizer* of ξ to be

$$G_\xi = \{g \in G \mid Ad_g(\xi) = \xi\} \subset G, \tag{3.4.6}$$

where Ad is the adjoint action of G on $\mathcal{G}$ defined in (3.1.7). G_ξ is actually a subgroup of G, and thus defines an action on Q. We have the following lemma.

LEMMA 3.4.5

    V_ξ is invariant under the action of G_ξ on Q, i.e.

44

$$V_\xi(\Phi_g(x)) \;=\; V_\xi(x), \quad \forall\; g \in G_\xi. \tag{3.4.7}$$

Proof

By a similar argument as in the proof of Lemma 3.2.4, we have

$$
\begin{aligned}
V_\xi(g \cdot x) \;=\;& V(g \cdot x) \;-\; K(g \cdot x)(Y(g \cdot x),\; \xi_Q(g \cdot x)) \\
& -\; \frac{1}{2} K(g \cdot x)(\xi_Q(g \cdot x),\; \xi_Q(g \cdot x)), \\
=\;& V(x) \;-\; K(x)(T\Phi_g^{-1} \cdot Y(g \cdot x),\; T\Phi_g^{-1} \cdot \xi_Q(g \cdot x)) \\
& -\; \frac{1}{2} K(x)(T\Phi_g^{-1} \cdot \xi_Q(g \cdot x), T\Phi_g^{-1} \cdot \xi_Q(g \cdot x)),
\end{aligned}
$$

since K is G-invariant. Moreover, for $g \in G_\xi$,

$$
\begin{aligned}
T\Phi_g^{-1} \cdot \xi_Q(g \cdot x) \;=\;& T\Phi_g^{-1} \cdot [\, \Phi_{exp\; \epsilon\xi} \cdot \Phi_g(x)\,], \\
=\;& [\, \Phi_{g^{-1} \cdot exp\; \epsilon\xi \cdot g}(x)\,] \;=\; [\, \Phi_{exp\; \epsilon\; Ad_{g^{-1}}\xi}(x)\,], \\
=\;& \xi_Q(x).
\end{aligned}
$$

With the above identity and the invariance property of Y, the lemma is proved.

∎

We assume that the quotient space Q/G_ξ is well defined. Denote the projection from Q to Q/G_ξ by π_ξ. By Lemma 3.4.5, we could define an induced function $\tilde{V}_\xi$ on Q/G_ξ from the augmented potential such that the diagram in Figure 3.4.1 commutes, namely,

$$\tilde{V}_\xi \;\circ\; \pi_\xi \;=\; V_\xi. \tag{3.4.8}$$
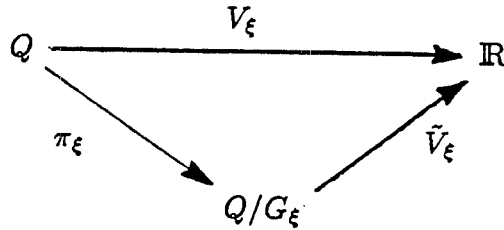
Figure 3.4.1. Symmetry of V_ξ

Typically, $\tilde{V}_\xi$ is a Morse function on Q/G_ξ and $\pi_\xi^{-1}(\tilde{x}_e)$ is a nondegenerate critical manifold in the sense of Bott[12], if $\tilde{x}_e$ is a critical point of $\tilde{V}_\xi$.

EXAMPLE 3.4.6

One application of the principle here is to find the relative equilibria of the planar three-body system discussed in [65] [66]. If we plot the function $\tilde{V}_\xi$ (for particular kinematic parameters) on the joint space, we get the picture in Figure 3.4.2, from which the *fundamental equilibria* defined in [65] [66]can be easily seen. These are the relative (joint) configurations $(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π).
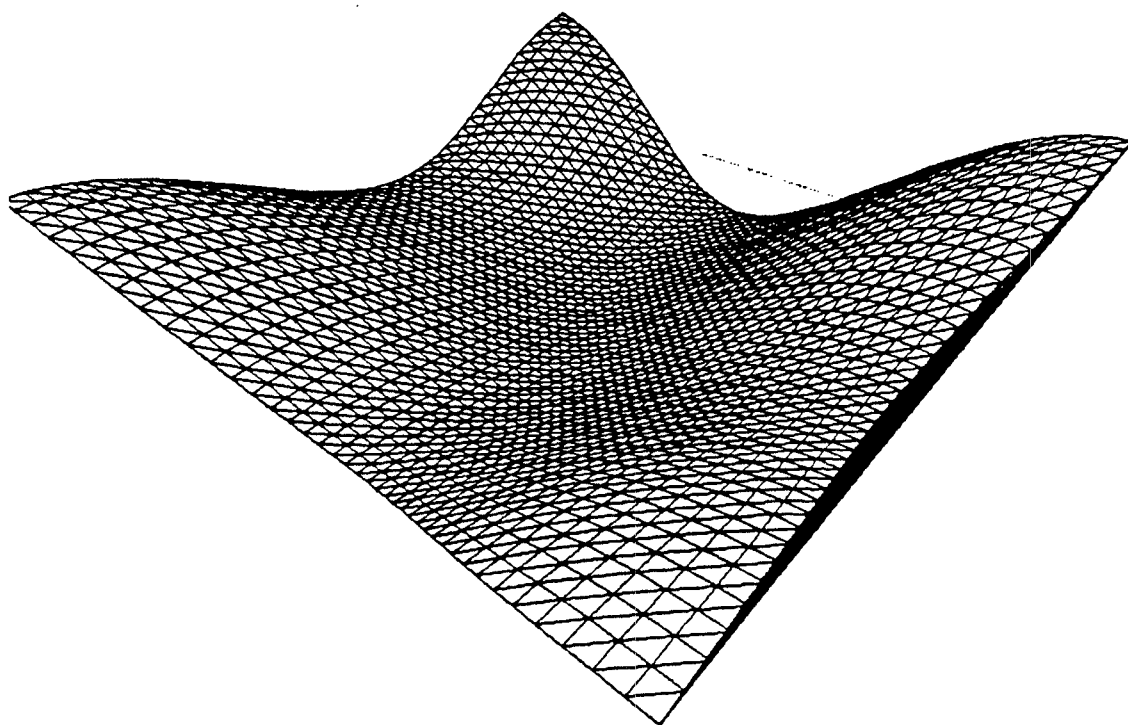
Figure 3.4.2. Function $\tilde{V}_\xi$ for the planar 3-body problem

As remarked at the end of Section 3.3, the reduction could be worked out on the Hamiltonian side. Thus there is a similar algorithm corresponding to Algorithm 3.4.4 on the T^*Q side. We include it here without derivation, which could be easily done in

46

an analogous way.

ALGORITHM 3.4.7

0. Pick $\xi \in \mathcal{G}$.

1. Search for the critical points x_e of the function

$$V_\xi : Q \to R$$
$$V_\xi(x) = V(x) - \ll Y(x),\ \xi_Q(x) \gg_x\ -\ \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x.$$

2. Find the corresponding conjugate momentum variable p_e, by putting x_e in the following formula.

$$p_e = K^\flat \left(Y(x_e) - \xi_Q(x_e) \right). \tag{3.4.9}$$

∎

The point $(x_e,\ p_e)$ in the momentum phase space T^*Q is then a relative equilibrium corresponding to the reduction on T^*Q with respect to the cotangent lift action.

3.5. Two Rigid Bodies Connected By a Ball-In-Socket Joint

The previous section provides us with a variational principle on configuration space for determining relative equilibria. In this section, we apply the principle of symmetric criticality to a problem of coupled rigid bodies. For simplicity, we consider here only a *simple mechanical system with symmetry*. Later in Chapter 7, the gyroscopic systems with symmetry will be discussed further. This section closely follows the work in [70]. A similar example was considered in [52], [53].

First we describe the kinematics of a mechanical system consisting of two rigid bodies connected by a spherical joint. Two bodies, with masses m_1, m_2, are free to move in three dimensional Euclidean space, subject to a (three-degrees-of-freedom) ball and socket coupling (See Figure 3.5.1). We introduce the following notations.
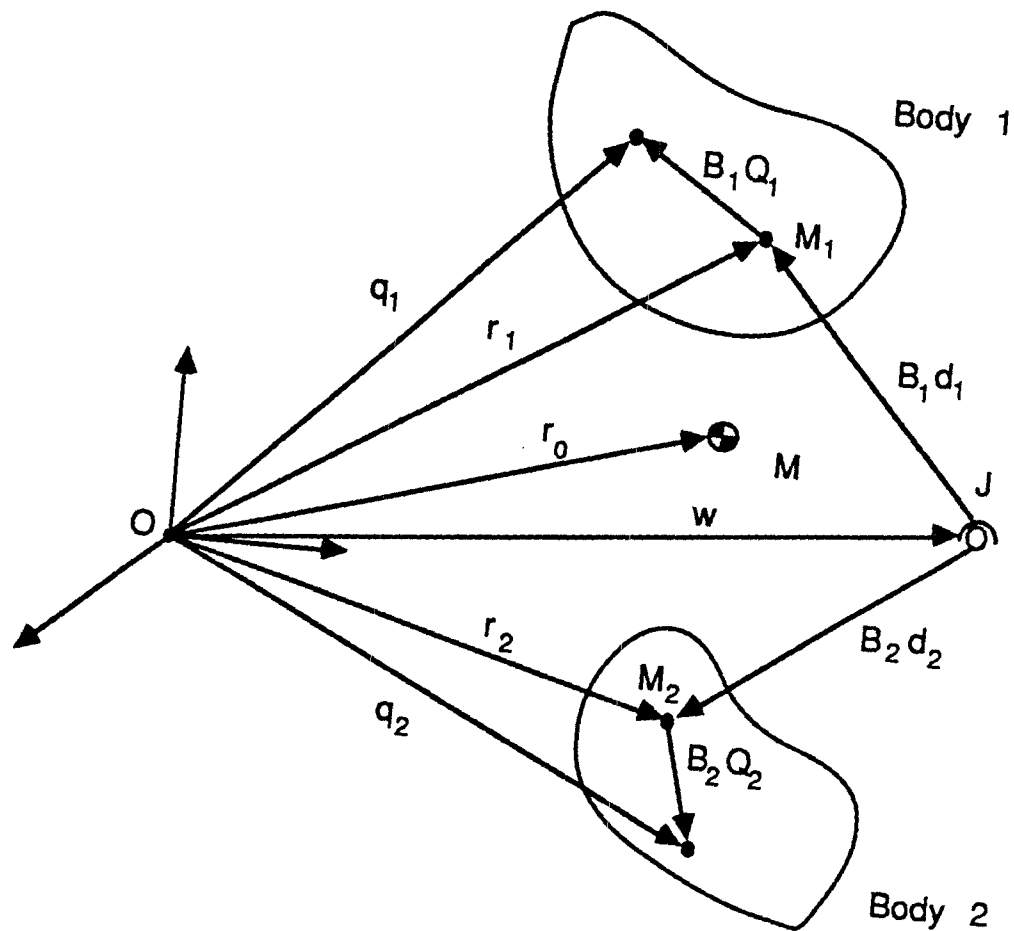
Figure 3.5.1. Rigid Bodies connected by a Ball-in-Socket Joint

Γ_0 : inertial frame of reference in space.

O : origin of the inertial reference system.

M_1 : center of mass of body 1.

M_2 : center of mass of body 2.

Γ_1 : orthonormal frame on body 1 with origin at M_1.

B_1 : rotational coordinate transformation matrix from Γ_1 to Γ_0.

Γ_2 : orthonormal frame on body 2 with origin at M_2.

B_2 : rotational coordinate transformation matrix from Γ_2 to Γ_0.

d_1 : vector from the joint to M_1 in the frame Γ_1.

d_2 : vector from the joint to M_2 in the frame Γ_2.

r_1: vector from O to M_1 in frame Γ_0.

r_2: vector from O to M_2 in frame Γ_0.

r_0: vector from O to the system center of mass in frame Γ_0.

m: total mass ($= m_1 + m_2$).

Q_1: vector from M_1 to a point of body 1 in the frame Γ_1.

q_1: vector from O to the same point of body 1 as Q_1 above in the frame Γ_0.

Q_2: vector from M_2 to a point of body 2 in the frame Γ_2.

q_2: vector from O to the same point of body 2 as Q_2 above in the frame Γ_0.

w: vector from O to the joint in the frame Γ_0.

From the above descriptions, we have the following kinematic relations,

$$q_1 = r_1 + B_1 Q_1, \tag{3.5.1a}$$

$$q_2 = r_2 + B_2 Q_2, \tag{3.5.1b}$$

$$m r_0 = m_1 r_1 + m_2 r_2, \tag{3.5.1c}$$

$$r_1 = w + B_1 d_1, \tag{3.5.1d}$$

$$r_2 = w + B_2 d_2. \tag{3.5.1e}$$

Also we know that B_1 and B_2 belong to the special orthogonal group $SO(3)$.

Let $\mu_1(\cdot)$ denote the mass measure of body 1 in the frame Γ_1 and $\mu_2(\cdot)$ denote the mass measure of body 2 in the frame Γ_2. The kinetic energy of body 1 can be thus written as

$$T_1 = \frac{1}{2} \int_{\mathcal{B}_1} |\dot{q}_1(Q_1)|^2 d\mu_1(Q_1).$$

Expanding the above by using (3.5.1a), (3.5.1b) and the formula $|x|^2 = tr(xx^T)$, we have the form

$$T_1 = \frac{m_1}{2} |\dot{r}_1|^2 + \frac{1}{2} tr(\dot{B}_1 I_1 \dot{B}_1^T).$$

where I_1 is the coefficient of inertia of body 1, defined by

$$I_1 \triangleq \int_{\mathcal{B}_1} Q_1 Q_1^T d\mu_1(Q_1),$$

and $tr(\cdot)$ denotes the trace of a matrix.

The kinetic energy of body 2 has a similar form. We thus have the total kinetic energy expressed as

$$
\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{m_1}{2} |\dot{r}_1|^2 + \frac{1}{2} tr(\dot{B}_1 I_1 \dot{B}_1^T) + \frac{m_2}{2} |\dot{r}_2|^2 + \frac{1}{2} tr(\dot{B}_2 I_2 \dot{B}_2^T).
\end{aligned}
\tag{3.5.2}
$$

By (3.5.1c)–(3.5.1e), we may write the total kinetic energy in terms of the total linear momentum $p = m\dot{r}_0$ of the system.

$$
T = \frac{1}{2} tr(\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} tr(\dot{B}_2 I_2 \dot{B}_2^T) + \frac{\varepsilon}{2} |\dot{B}_1 d_1 - \dot{B}_2 d_2|^2 + \frac{1}{2m} |p|^2.
$$

Here $\varepsilon \triangleq m_1 m_2/(m_1 + m_2)$ is the reduced mass. Since there is no potential assumed, this is also the Lagrangian of the system.

The configuration space is $SO(3) \times SO(3) \times \mathbb{R}^3$. The system is invariant under translation of the inertial reference frame, i.e. we have a symmetry group action on the configuration space

$$
\Phi: \mathbb{R}^3 \times \big(SO(3) \times SO(3) \times \mathbb{R}^3\big) \to SO(3) \times SO(3) \times \mathbb{R}^3
$$

$$
(\lambda, (B_1, B_2, r)) \mapsto (B_1, B_2, \lambda + r).
$$

We can symplectically reduce the system by $\mathbb{R}^3$ (see [48], [2]) which in turn corresponds to jumping to the center of the inertial frame. This is also done in [27] and for planar problem in [50][65][66]. After this reduction, the reduced Lagrangian is

$$
L = \frac{1}{2} tr(\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} tr(\dot{B}_2 I_2 \dot{B}_2^T) + \frac{\varepsilon}{2} |\dot{B}_1 d_1 - \dot{B}_2 d_2|^2.
\tag{3.5.3}
$$

which is a function on $T(SO(3) \times SO(3))$.

Although the mechanical system considered here is exactly the same as in [27], the Lagrangian is expressed in terms of coefficients of inertia referred to different body frames than the one they use. Ours is based on the body frames affixed to centers of mass. By applying the formula for change of coefficient of inertia by translation, one checks that the results are the same.

Now we put the system in the category of simple mechanical system with symmetry. The Riemannian metric on $T(SO(3) \times SO(3))$ is given by the (symplectically) reduced Lagrangian as

$$\ll (W_1, W_2), (W_1, W_2) \gg = tr(W_1 I_1 W_1^T) + tr(W_2 I_2 W_2^T)$$

$$\varepsilon \, | \, W_1 d_1 - W_2 d_2 \, |^2 .$$

where (W_1, W_2) belongs to $T(SO(3) \times SO(3))$. We know that every element in $T(SO(3) \times SO(3))$ can be represented as in, cf. Section 2.2,

$$T(SO(3) \times SO(3)) = \{ (B_1, B_2, \hat{w}_1 B_1, \hat{w}_2 B_2) \, :$$

$$B_1, B_2 \in SO(3), \; w_1, w_2 \in \mathbb{R}^3 \},$$

where $\hat{} : \mathbb{R}^3 \to so(3)$ is defined in (2.2.1). In terms of w_1, w_2, we have

$$\ll (W_1, W_2), (W_1, W_2) \gg \; = \; tr(\hat{w}_1 B_1 I_1 B_1^T \hat{w}_1^T) + tr(\hat{w}_2 B_2 I_2 B_2^T \hat{w}_2^T)$$

$$+ \; \varepsilon \, | \, \hat{w}_1 B_1 d_1 - \hat{w}_2 B_2 d_2 \, |^2 .$$

As shown in (2.2.2e), we have the following relation,

$$tr(\hat{\omega} \, I \, \hat{\omega}^T) \; = \; < \omega, \, \mathbf{I}\omega >_E, \tag{3.5.4}$$

where $<, >_E$ is the Euclidean inner product, and, with the physical interpretation, I is the *coefficient of inertia* tensor and $\mathbf{I}$ is the associated *moment of inertia* tensor related by the formula in (2.2.2e). Upon further simplifications and rearrangements, we get

$$\ll (W_1, W_2), (W_1, W_2) \gg = ((B_1^T w_1)^T \; (B_2^T w_2)^T) \begin{pmatrix} \mathbf{J}_1 & \mathbf{J}_{12} \\ \mathbf{J}_{12}^T & \mathbf{J}_2 \end{pmatrix} \begin{pmatrix} B_1^T w_1 \\ B_2^T w_2 \end{pmatrix}, \tag{3.5.5a}$$

where

$$\mathbf{J}_1 \; = \; \mathbf{I}_1 \; + \; \varepsilon \, \hat{d}_1^T \hat{d}_1$$

$$\mathbf{J}_2 \; = \; \mathbf{I}_2 \; + \; \varepsilon \, \hat{d}_2^T \hat{d}_2 \tag{3.5.5b}$$

$$\mathbf{J}_{12} \; = \; \varepsilon \, \hat{d}_1 B_1^T B_2 \hat{d}_2$$

The group action to consider is defined on $SO(3) \times SO(3)$, the configuration space, *relative to an observer at the system center of mass.* The diagonal action of the group $G = SO(3)$ is given by

$$\Psi : G \times (SO(3) \times SO(3)) \to SO(3) \times SO(3)$$

$$(R, (B_1, B_2)) \mapsto (RB_1, RB_2). \tag{3.5.6}$$

Letting $\hat{\xi} \in \mathcal{G}$, the corresponding infinitesimal generator can be found as

51

$$\xi_Q(B_1, B_2) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\exp \epsilon \,\hat{\xi})(B_1, B_2)$$

$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left((\exp \epsilon \,\hat{\xi})B_1, (\exp \epsilon \,\hat{\xi})B_2\right) \qquad (3.5.7)$$

$$= (\hat{\xi}B_1, \hat{\xi}B_2).$$

Since here the potential energy V is identically 0, and the gyroscopic field vanishes, i.e. $Y = 0$, the function V_ξ, cf. (3.4.5), is

$$V_\xi(B_1, B_2) = -\frac{1}{2}\left((B_1^T\xi)^T \quad (B_2^T\xi)^T\right)\begin{pmatrix} \mathbf{J}_1 & \mathbf{J}_{12} \\ \mathbf{J}_{12}^T & \mathbf{J}_2 \end{pmatrix}\begin{pmatrix} B_1^T\xi \\ B_2^T\xi \end{pmatrix}. \qquad (3.5.8)$$

It is clear that V_ξ is invariant under $G_\xi = \{R \in G : R\xi = \xi\}$, cf. (3.4.6),(3.4.7), which is isomorphic to S^1.

By the compactness of $SO(3) \times SO(3)$ we know that for each ξ, V_ξ has critical points. We need to find the conditions on B_1, B_2 so that the gradient of V_ξ with respect to B_1, B_2 is 0. Equivalently one can check the vanishing of the differential dV_ξ on the space $T(SO(3) \times SO(3))$. Let $W \in T(SO(3) \times SO(3))$,

$$W = (B_1, B_2, \hat{w}_1 B_1, \hat{w}_2 B_2).$$

The curve in $SO(3) \times SO(3)$ generated by W is $(e^{t\hat{w}_1}B_1, e^{t\hat{w}_2}B_2)$. Thus we have the formula

$$dV_\xi(B_1, B_2)(W) = \frac{d}{dt}\bigg|_{t=0} V_\xi(e^{t\hat{w}_1}B_1, e^{t\hat{w}_2}B_2),$$

Explicitly, we get the following final form (here $B = B_1^T B_2$),

$$dV_\xi(B_1, B_2) \cdot W =$$

$$< w_1, \hat{\xi}B_1 \mathbf{J}_1 B_1^T \xi >_E + < w_2, \hat{\xi}B_2 \mathbf{J}_2 B_2^T \xi >_E$$

$$+ \varepsilon < w_1, \hat{\xi}B_1 \hat{d}_1 B \hat{d}_2 B_2^T \xi >_E + \varepsilon < w_2, \hat{\xi}B_2 \hat{d}_2 B^T \hat{d}_1 B_1^T \xi >_E$$

$$+ \varepsilon < w_1, \widehat{B_1 \hat{d}_1 \xi}\, B_2 \hat{d}_2 B_2^T \xi >_E$$

$$+ \varepsilon < w_2, \widehat{B_2 \hat{d}_2 \xi}\, B_1 \hat{d}_1 B_1^T \xi >_E$$

Thus we know that the necessary conditions for a critical point of V_ξ are

$$\hat{\xi}B_1 \mathbf{J}_1 B_1^T \xi + \varepsilon\, \hat{\xi}B_1 \hat{d}_1 B \hat{d}_2 B_2^T \xi + \varepsilon\, \widehat{B_1 \hat{d}_1 \xi}\, B_2 \hat{d}_2 B_2^T \xi = 0,$$

$$\hat{\xi}B_2 \mathbf{J}_2 B_2^T \xi + \varepsilon\, \hat{\xi}B_2 \hat{d}_2 B^T \hat{d}_1 B_1^T \xi + \varepsilon\, \widehat{B_2 \hat{d}_2 \xi}\, B_1 \hat{d}_1 B_1^T \xi = 0.$$

Now if we define $\Omega_1 \equiv B_1^T \xi$, and $\Omega_2 \equiv B_2^T \xi$, we get the conditions (in terms of cross products in $\mathbb{R}^3$)

$$\Omega_1 \times \mathbf{J}_1 \Omega_1 + \varepsilon\, d_1 \times (\Omega_1 \times B(d_2 \times \Omega_2)) = 0, \tag{3.5.9a}$$

$$\Omega_2 \times \mathbf{J}_2 \Omega_2 + \varepsilon\, d_2 \times (\Omega_2 \times B^T(d_1 \times \Omega_1)) = 0, \tag{3.5.9b}$$

which are exactly the conditions found by Poisson reduction in [unpublished notes of P.S. Krishnaprasad]. In step 2 of the Algorithm 3.4.4 in Section 3.4, we put in the B_1, B_2 found by solving the above conditions into

$$v = \xi_Q(B_1, B_2).$$

Let $v \in T_{(B_1, B_2)}(SO(3) \times SO(3))$ be represented as

$$v = (\hat{\omega}_1 B_1, \hat{\omega}_2 B_2).$$

We find that ω_1, ω_2 is nothing but

$$\omega_1 \;=\; \omega_2 \;=\; \xi.$$

The conjugate momentum variables at relative equilibrium could be found from (3.4.9) as follows. Let $p \in T^*_{(B_1, B_2)}(SO(3) \times SO(3))$ be represented as

$$p = (\hat{\alpha}_1 A_1, \hat{\alpha}_2 A_2).$$

We find that α_1, α_2 can be expressed as

$$\alpha_1 = A_1 \mathbf{J}_1 \Omega_1 + \varepsilon\, (A_1 d_1 \times A_2(d_2 \times \Omega_2)),$$

$$\alpha_2 = A_2 \mathbf{J}_2 \Omega_2 + \varepsilon\, (A_2 d_2 \times A_1(d_1 \times \Omega_1)).$$

Now we derive a necessary geometric condition for relative equilibria. The relation between Ω_1 and Ω_2 is

$$\Omega_1 = B\Omega_2.$$

If we let $s_1 = B_1 d_1$, $s_2 = B_2 d_2$, from (3.5.9a), we get

$$B_1^T \xi \times \mathbf{J}_1 B_1^T \xi + \varepsilon\, B_1^T s_1 \times (B_1^T \xi \times B(B_2^T s_2 \times B_2^T \xi)) = 0,$$

53

which implies

$$\xi \times B_1 J_1 B_1^T \xi + \varepsilon\, s_1 \times (\xi \times (s_2 \times \xi)) = 0. \tag{3.5.10}$$

Taking the inner product of (3.5.10) with ξ, we obtain a key necessary condition for a relative equilibrium

$$\xi \cdot (s_1 \times s_2) = 0. \tag{3.5.11}$$

We note that ξ is the axis of rotation of the whole body, s_1, s_2 are the spatial vectors from joint to body 1 and 2, respectively. From (3.5.11), we conclude that, at relative equilibria, ξ, s_1, s_2 *must lie on the same plane, no matter what the inertias are.*

REMARK 3.5.1

We note without proof that $\Omega_1 = B_1 \xi$ and $\Omega_2 = B_2 \xi$ satisfy two additional conditions:

(a) Ω_1 is an eigenvector of $\mathbf{I}_{lock}$,

(b) Ω_2 is an eigenvector of $B^T \mathbf{I}_{lock}\, B$, where

$$\mathbf{I}_{lock} = \mathbf{J}_1 + B J_2 B^T + B J_{12}^T + \mathbf{J}_{12} B^T \tag{3.5.12}$$

is the locked inertia dyadic of the system of two bodies referred to the body 1 frame. ∎

Although we can get the same critical conditions (3.5.9a) (3.5.9b) by other methods, the principle of symmetric criticality provides more information. Notice that in the first step of the Algorithm 3.4.4, we simply try to find the critical points of V_ξ on $SO(3) \times SO(3)$ without any additional constraint. Thus one has an associated unconstrained optimization problem. Numerical optimization schemes can be used to find relative equilibria of minimum or maximum type. This issue is discussed in the following.

By the symmetry of the system, we know that the function V_ξ is invariant in the direction tangent to the orbit of G_ξ. Thus in the search for critical points, we should avoid these directions. It turns out that the usual gradient-type method is a good choice. Here, we use an optimization package named CONSOLE which was developed at the

University of Maryland[20]. The current version of CONSOLE basically uses the steepest descent method and is thus applicable to our circumstances.

In formulating the optimization problem, in order to avoid other constraints arising from the restrictions on $SO(3)$, e.g. $A^T A = Identity$, we use Cayley's parametrization. That is, any element $A \in SO(3)$ can be represented by

$$A = \frac{1}{1 + a_1^2 + a_2^2 + a_3^2} \begin{pmatrix} 1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1 a_2 - a_3) & 2(a_1 a_3 + a_2) \\ 2(a_1 a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2 a_3 - a_1) \\ 2(a_1 a_3 - a_2) & 2(a_2 a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$
$$(3.5.13)$$

where $a_1, a_2, a_3 \in R$. The problem can now be written as

$$\begin{array}{cc} extremize & V_\xi(B_1, B_2) \\ \left\{ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{array} \right\} \end{array}$$

where (a_1, a_2, a_3), (b_1, b_2, b_3) are the parameters for B_1, B_2, respectively.

The CAD package CONSOLE is composed of two main programs: CONVERT, SOLVE. CONVERT reads a problem description file which describes the optimization problem to be solved. SOLVE then performs the optimization process with the interaction of user and/or some simulator. For more details, see Fan et el. [21], [19]. The problem description file for our problem is easily formulated in Table 3.5.2.

Table 3.5.2. Problem Description File

```
design_parameter a1 init=0
design_parameter a2 init=0
design_parameter a3 init=0

design_parameter b1 init=0
design_parameter b2 init=0
design_parameter b3 init=1

objective "V-xi"
   minimize {
      import a1, a2, a3;
      import b1, b2, b3;

      double cost();

      return  cost( a1, a2, a3, b1, b2, b3 );
      }
   good_value=0
   bad_value=100
```

55

where the subroutine `cost()` reads a system description file containing the information of I_1, I_2, d_1, d_2, ξ, m_1, m_2 and then returns the value of the function V_ξ. By choosing different moments of inertia and initial structure, we can perform the optimization. In the process, one thing we learned is that if the augmented inertia is diagonal, the rate of convergence is faster. Thus preliminary diagonalizations should be performed to get speed up.

In the particular case that

$$m_1 = 3.0, \quad m_2 = 2.0,$$

$$d_1 = (0\ 0\ 1), \quad d_2 = (-1\ 1\ 1), \quad \xi = (0\ 0\ 1),$$

$$I_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

the relative equilibrium we found by numerical methods was

$$B_1 = \begin{pmatrix} 0.0 & -0.939 & 0.344 \\ 0.0 & -0.344 & -0.939 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.007 & 0.350 & -0.937 \\ -0.528 & 0.796 & 0.294 \\ 0.849 & 0.493 & 0.191 \end{pmatrix},$$

$$s_1 = (0.344\ -0.939\ 0.0), \quad s_2 = (-0.593\ 1.618\ -0.165),$$

$$\Omega_1 = (1.0\ 0.0\ 0.0), \quad \Omega_2 = (0.849\ 0.493\ 0.191).$$

Several relative equilibria corresponding to different choices of parameters are shown in Figure 3.5.3. Case 1 in that figure corresponds to the above numerical result.

Note that in all cases, either $s_1 = B_1 d_1$, $s_2 = B_2 d_2$ are on a straight line or they and ξ are on one plane. It matches with the condition (3.5.11).

Case 1

ξ   [0 0 1]

[0.344 -0.939 0]

$B_1 d_1$

$B_2 d_2$

[-0.593 1.6182 -0.1646]

Case 2

ξ   [0 0 1]

$B_1 d_1$

[-1.3966 -0.1311 0.1796]

$B_2 d_2$

[2.2134 0.2081 -0.2397]

Case 3

ξ   [0 0 1]

[-1.2868 0.5866 0]

[1.2868 -0.5866 0]

$B_2 d_2$

$B_1 d_1$
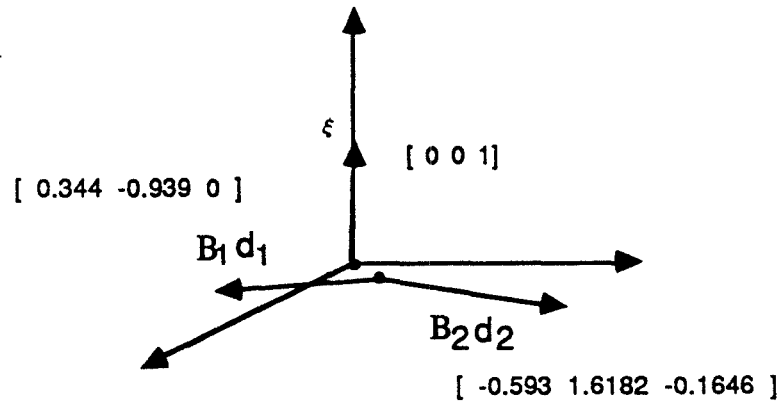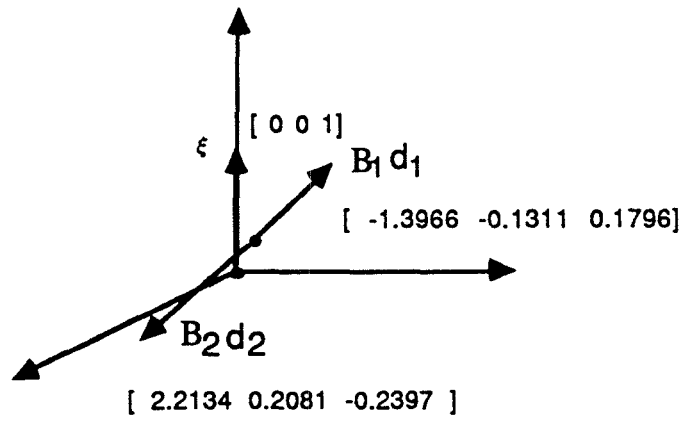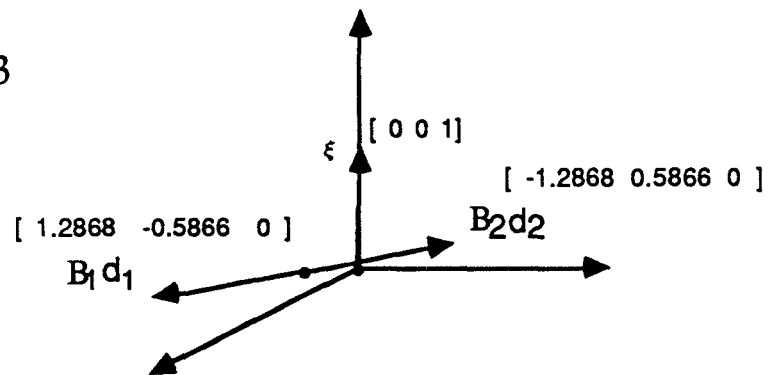
Figure 3.5.3. Relative Equilibria Configurations

57

CHAPTER IV

Hamiltonian and Dissipative Systems

Among the most important dynamical systems encountered in physical sciences are the conservative systems (including the hamiltonian systems) and those exhibiting some types of dissipation [57] [29]. In this chapter, we discuss how dissipation may enter into a hamiltonian system and how it can affect the dynamical behavior. Typically damping mechanisms drive a system asymptotically to a stable equilibrium state. However, it is not necessary to put damping at every interconnection. Partial damping may either damp the system out or drive the system into an nontrivial motion. These issues will be addressed in the following. A closely related work can be found in [71].

4.1. Hamiltonian Systems with Added Dissipation

In this section, we will see how dissipation enters a mechanical system through exterior forces (or horizontal 1-forms). Let P be a Poisson manifold with a Poisson structure $\{,\}$. Let H be a smooth function on P and X_H be the corresponding Hamiltonian vector field as defined in an analogous way as in (3.3.7), or

$$X_H[f] = \{f, H\}, \quad \forall f \in C^\infty(P). \tag{4.1.1}$$

Recall that vector fields can be thought as derivations on the space of C^∞ functions on P, which will be denoted by $\mathcal{F}(P)$. The derivation on the function f corresponding to a vector field X is usually written as $L_X f$ or simply $X[f]$, which is again an element in $\mathcal{F}(P)$. Now let us consider a dynamical system which could be written in the form of

$$\dot{p} = X_H(p) + X^D(p), \tag{4.1.2}$$

where $p \in P$, and X^D is a vector field on P. We have the following definition.

<u>DEFINITION 4.1.1</u>

A vector field X^D is called a *dissipative field* with respect to the Hamiltonian system X_H in the region $O \subset P$, if

(i) $X^D[H](p) \leq 0, \quad \forall\, p \in O$.

(ii) For $x \in O$,

$$X^D[H](p) = 0, \quad \text{iff} \quad X^D(p) = 0.$$

∎

Condition (i) says that the directional derivative of H along X^D is non-positive, or the Hamiltonian (energy) decays along the direction of X^D. The second condition tells us that wherever the decay vanishes, the dissipative field must vanish as well.

Now we discuss a special type of dissipative fields which are induced by exterior forces in (regular) lagrangian mechanics. We consider the case of $P = TQ$ with the Poisson structure (3.3.4). Recall that Lagrange-d'Alembert principle for the system without constraints gives us the formula (2.1.14),

$$F_L(X) + \omega = 0,$$

which could be written as

$$\Pi_L^{-1}(X) - dH_L + \omega = 0, \tag{4.1.3a}$$

or

$$X - \Pi_L(dH_L) + \Pi_L(\omega) = 0. \tag{4.1.3b}$$

If we let

$$X^D \triangleq -\Pi_L(\omega), \tag{4.1.4}$$

then the virtual displacement corresponding to the dynamical motion is

$$X = X_{H_L} + X^D. \tag{4.1.5}$$

We define the following notion.

DEFINITION 4.1.2

The exterior force ω is called a *dissipative force* for a lagrangian system with Lagrangian L in the subbundle $O \subset TQ$ if

$$\omega(X_{H_L})(v_x) \leq 0, \quad \forall\, v_x \in O$$

and $\omega = 0$ on the zero section of O, namely,

$$\omega(x,0) = 0, \quad \text{for } (x,0) \in O.$$

∎

Recall from Lemma 2.1.5, the Lagrangian vector field X_{H_L} is a special vector field on TQ, namely in local coordinates, $X_{H_L}(x,v) = (v,\ w)$ for $w \in T_x Q$. Since ω is a horizontal 1-form on TQ, in local coordinates, we may write $\omega(x,v) = (\alpha,\ 0)$, where $\alpha \in T_x^* Q$. It follows that

$$\omega(X_{H_L})(x,v) = \langle \alpha,\ v \rangle. \qquad (4.1.6)$$

The function $\omega(X_{H_L})$ on TQ is thus independent of the Lagrangian L. Consequently, we may define a dissipative force for any lagrangian system as follows.

DEFINITION 4.1.3

The exterior force ω is called a *dissipative force* for lagrangian systems in the subbundle $O \subset TQ$ if, in local coordinates, for $\omega(x,v) = (\alpha(x,v),\ 0)$, we have the following properties.

$$\langle \alpha(x,v),\ v \rangle \leq 0, \quad \forall\, (x,v) \in O,$$

$$\alpha(x,0) = 0, \quad \forall\, (x,0) \in O.$$

∎

Namely, α must vanish on the zero-section of the bundle. For the case that α is in the form of

$$\alpha(x,v) = \gamma \cdot v, \qquad (4.1.7)$$

where γ is a negative definite matrix, then

$$\langle \alpha, \ v \rangle \ = \ v \cdot \gamma \cdot v, \tag{4.1.8}$$

which is exactly the Rayleigh's dissipation function considered in the literature [23].

In the setting of lagrangian mechanics, the Lagrangian vector field X_{H_L} has a special character, namely, it is a *special vector field*, cf. Lemma 2.1.5. Also the virtual displacement X must be a special vector field, cf. (4.1.5). Thus to define a dissipative field for such lagrangian system, we need one additional assumption.

DEFINITION 4.1.4

A vector field X^D is called a *dissipative field* for a lagrangian system X_{H_L} in the subbundle $O \subset TQ$, if

(i) $X^D[H_L](v_x) \ \leq \ 0, \quad \forall \ v_x \in O$.

(ii) For $x \in O$,

$$X^D[H_L](v_x) \ = \ 0, \quad \text{iff} \quad X^D(v_x) \ = \ 0.$$

(iii) X^D is a vertical vector field.

∎

It is then natural to state the following fact.

LEMMA 4.1.5

The vector field associated with a dissipative force, defined in (4.1.4), is a dissipative field for any lagrangian system.

Proof

Since the map Π_L maps any horizontal 1-form to a vertical vector field, cf. Lemma 2.1.4, condition (iii) is automatic. The lagrangian system can be written in the form of (4.1.5). It follows that

$$X^D[H_L] \ = \ dH_L(X^D) \ = \ -dH_L(\Pi_L(\omega)),$$

$$= \ -\Omega_L(\Pi_L(dH_L), \ \Pi_L(\omega)) \ = \ \Omega_L(\Pi_L(\omega), \ X_{H_L})$$

$$= \ \omega(X_{H_L}). \tag{4.1.9}$$

It is then easily seen that the vector field X^D defined in (4.1.4) is a dissipative field.

∎

For a simple mechanical system, the Lagrangian is

$$L(v_x) = \frac{1}{2} \ll v_x, \; v_x \gg_x \; - V(x),$$

with the energy function

$$H_L(v_x) = \frac{1}{2} \ll v_x, \; v_x \gg_x \; + V(x).$$

Since dV is a horizontal 1-form, it annihilates any vertical vector field. Thus we only need to check conditions (i), (ii) in Definition 4.1.4 with the kinetic energy, or riemannian metric, instead of H_L. This is the definition adopted in [2], p. 234, cf. also [60], where the dissipative vector field is defined through conditions (i), (iii) with replacing the total energy by the kinetic energy. For a gyroscopic system, since the energy function is not affected by including the gyroscopic term, the same remark applies.

4.2. LaSalle Invariance Principle and Maximal Invariant Set

In this section, we review the LaSalle Invariance Principle and discuss various ways to characterize the maximal invariant set. We consider the following system,

$$\dot{x} = X(x), \quad x \in Q, \, X \in \mathfrak{X}(Q). \tag{4.2.1}$$

The flow of X will be denoted by Φ^t_X which is assumed to be a diffeomorphism on Q. $\Phi^t_X(x_0)$ denotes the point in Q of the flow of X at time t which starts from x_0 at time 0. We only consider the case $t \geq 0$ here. We have the following notion.

DEFINITION 4.2.1

A set $\mathcal{I}$ in Q is called an *invariant set* for the system (4.2.1) if, for every $y \in \mathcal{I}$,

$$\Phi^t_X(y) \in \mathcal{I}, \quad \forall \, t \geq 0. \tag{4.2.2}$$

Let S be a subset in Q, $\mathcal{I}$ is called the *maximal invariant set* in S if $\mathcal{I}$ is an invariant set and if every invariant set in S is contained in $\mathcal{I}$.

∎

Equivalently, by a straightforward argument, we may describe the maximal invariant set in S as the set of all points $y \in S$ such that

$$\Phi_X^t(y) \in S, \quad \forall \, t \geq 0. \tag{4.2.3}$$

Accordingly, to find the maximal invariant set in S, we need to search for all the points $y \in S$, such that the solution of

$$\dot{x} = X(x),$$

$$x(0) = y,$$

stays in S forever. Since the solution $x(t)$ stays in S for all t, we could restrict the vector field X to the set S and consider the system,

$$\dot{x} = X\big|_S (x),$$

$$x(0) = y \in S. \tag{4.2.4}$$

If the solution for the system (4.2.4) always lies in S, then y must be a point in the maximal invariant set. On the other hand, every point in the maximal invariant set must generate a solution of (4.2.4). Correspondingly, we have the following lemma to characterize the maximal invariant set in S.

LEMMA 4.2.2

The maximal invariant set in S of (4.2.1) is the set of all points $y \in S$ such that the system (4.2.4) generates a curve in S.

∎

Now we consider the case that S is a level set of a smooth function f on Q, i.e.

$$S = \{ \, x \in Q \; : \; f(x) = c \, \},$$

where c is a constant in $\mathbb{R}$. From (4.2.3), it is easy to verify that the maximal invariant set in S could be written as

$$\mathcal{I} = \{ \, x \in S \; : \; f\big(\Phi_X^t(x)\big) = c, \, \forall \, t \geq 0 \, \}. \tag{4.2.5}$$

We assume that both the vector field X and the function f are analytic. We have the Lie series formula, [26],

$$f\left(\Phi_X^t(x)\right) = \sum_{k=0}^{\infty} L_X^k f(x) \frac{t^k}{k!}$$

$$= f(x) + \sum_{k=1}^{\infty} L_X^k f(x) \frac{t^k}{k!}. \qquad (4.2.6)$$

The maximal invariant set in S could be further written as, from (4.2.5), (4.2.6),

$$\mathcal{I} = \{ x \in S \ : \ f(x) = c, \ L_X^k f(x) = 0, \ \text{for } k = 1, 2, \dots \}. \qquad (4.2.7)$$

This formula provides us a convenient and systematic way to find the maximal invariant set. We next recall LaSalle's Theorem. [40]

THEOREM 4.2.3 (LaSalle Invariance Principle)

Let V be a smooth function on Q. Let Γ_c denote the region where $V(x) \leq c$. Assume that Γ_c is bounded and that within Γ_c, we have

$$V(x) > 0, \quad \text{for } x \neq 0, \quad \text{and } V(0) = 0,$$

$$L_X V(x) \leq 0. \qquad (4.2.8)$$

Let $\mathcal{R}$ be the set of points x within Γ_c, where $L_X V(x) = 0$, and let $\mathcal{M}$ denote the maximal invariant set in $\mathcal{R}$. Then every solution $x(t)$ in Γ_c tends to $\mathcal{M}$ as $t \to \infty$.

∎

A function satisfying (4.2.8) will be called a *Lyapunov function*, cf. [28]. Here, the set $\mathcal{R}$ could be written as

$$\mathcal{R} = \{ x \in \Gamma_c \ : \ L_X V(x) = 0 \}. \qquad (4.2.9)$$

For the analytic case, we could readily write the maximal invariant set $\mathcal{M}$ in $\mathcal{R}$ as, from (4.2.7),

$$\mathcal{M} = \{ x \in \Gamma_c \ : \ L_X^k V(x) = 0, \ \text{for } k = 1, 2, 3, \dots \}. \qquad (4.2.10)$$

From Lemma 4.2.2, we could also consider the system

$$\dot{x} = X|_{\mathcal{R}}(x),$$

$$x(0) = y \in \mathcal{R}. \qquad (4.2.11)$$

to find the maximal invariant set. Every trajectory starting in Γ_c will approach this set $\mathcal{M}$ as t goes to infinity.

Now we consider a hamiltonian system with added dissipation (4.1.2). The Hamiltonian is by construction a suitable Lyapunov function. From the Invariance Principle, we have the following theorem.

THEOREM 4.2.4

Let $\Gamma_c = \{\, p : H(p) \le c \,\} \subset O$. Assume that Γ_c is bounded. Then, as $t \to \infty$, every solution of (4.1.2) in Γ_c approaches the set,

$$\mathcal{M} = \{\, p \in \Gamma_c : L^k_{X_H} L_{X^D} H \,(p) = 0, \text{ for } k = 0,1,2,... \,\}. \tag{4.2.12}$$

Proof

We only need to verify the conditions in (4.2.10) are equivalent to the conditions in (4.2.12). In fact,

$$L_X H = L_{X_H + X^D} H = L_{X_H} H + L_{X^D} H = L_{X^D} H,$$

since H is a first integral along the vector field X_H. Thus we have the condition

$$L_{X^D} H(p) = 0.$$

But, by property (ii) in Definition 4.1.1, this implies that $X^D(p) = 0$. It follows that

$$L^2_X H(p) = L_{X_H} L_{X_D} H(p).$$

By induction, the set in (4.2.12) is equal to the set in (4.2.10).

∎

In terms of the Poisson structure, we could write the set in (4.2.12) as

$$\mathcal{M} = \{\, p \in \Gamma_c : L_{X^D} H(p) = 0, \{H, L_{X^D} H\}(p) = 0,$$
$$\{H, \{H, L_{X^D} H\}\}(p) = 0, ..., etc.\}. \tag{4.2.13}$$

For the lagrangian system with dissipative force, cf. (4.1.5), the energy function H_L is a Lyapunov function. For gyroscopic systems with symmetry, we can get a similar condition as in (4.2.12), just by replacing p by v_x, H by H_L. Moreover, from (4.1.9), as proved in Lemma 4.1.5, the maximal invariant set could be further written as

$$\mathcal{M} = \{\, v_x \in \Gamma_c : L^k_{X_{H_L}} \omega(X_{H_L}) \,(v_x) = 0, \text{ for } k = 0,1,2,... \,\}. \tag{4.2.14}$$

Also note that for this case the set $\mathcal{R}$, cf. (4.2.9) can be written as

$$\mathcal{R} = \{\ v_x \in \Gamma_c\ :\ \omega(X_{H_L})\,(v_x)\ =\ 0\ \}.$$

From (4.1.6), in local coordinates, we may simplify the expression as

$$\mathcal{R} = \{\ (x,v) \in \Gamma_c\ :\ \langle \alpha(x,v),\ v \rangle\ =\ 0\ \}. \tag{4.2.15}$$

Thus for a lagrangian system with dissipative force as defined in Definition 4.1.3, the set $\mathcal{R}$ is greatly simplified. With these conditions in $\mathcal{R}$, we may apply the techniques discussed before directly to find the maximal invariant set.

4.3. Multibody Analog of Dual-spin Problem

In this section, we apply the techniques developed before to a problem which is a multibody analog of the dual-spin problem. In the field of spacecraft attitude control and stabilization, the dual-spin maneuver has an important place. In designing a communication satellite or an interplanetary probe, engineers are often faced with the requirement that the spacecraft be able to maintain a fixed orientation relative to some inertial frame. The dual-spin technique is a simple, commonly used technique for meeting this requirement. A dual-spin spacecraft consists of the spacecraft body and on-board motor-driven symmetric rotors. In the presence of a suitable damping mechanism and for sufficiently high rotor velocities, the attitude acquisition can be achieved. The final state is a steady spin about a fixed axis. This single rigid body dual-spin problem has been studied extensively before, e.g. [36] [34] [14] and the references therein. A rigorous proof of asymptotic stability can be found in [36].

We consider a similar system as described in Section 3.5. But now we mount rotors on each body and obtain an assembly as shown in Figure 4.3.1. Here the reference inertial frame is placed at the center of mass of the assembly. This corresponds to reduction with respect to the translational invariance of the system, as discussed in Section 3.5. Ignoring the specific kinematic relationships between the rotors and the bodies, the unconstrained configuration space Q_u is parametrized by the attitudes of these eight bodies,

66
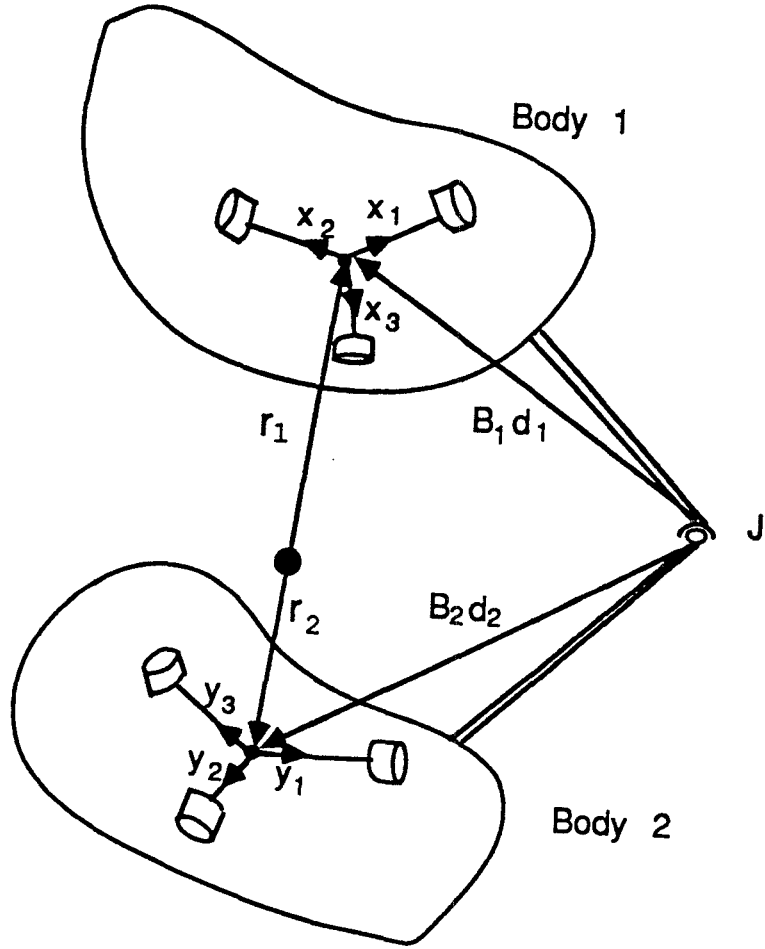
Figure 4.3.1. Two Rigid Bodies with Rotors

$$Q_u = \{ (B_1, \ S_1, \ S_2, \ S_3, \ B_2, \ D_1, \ D_2, \ D_3) \},$$
$$= SO(3) \times \left(SO(3)\right)^3 \times SO(3) \times \left(SO(3)\right)^3.$$

(4.3.1)

To account for the body-rotor relations, we have the following constraints between the attitudes,

$$S_i = B_1 R(x_i, \theta_i), \quad i = 1, 2, 3,$$

(4.3.2a)

$$D_i = B_2 R(y_i, \phi_i), \quad i = 1, 2, 3.$$

(4.3.2b)

where $R(x_i, \theta_i)$ is the rotation about the x_i axis by the angle θ_i, e.g.

$$R(x_3, \theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(4.3.3)

67

With these constraints, the configuration space is then

$$Q = SO(3) \times (S^1)^3 \times SO(3) \times (S^1)^3.$$

Assume, for convenience, the centers of mass of rotors $\mathcal{S}_i$ are at the center of mass of carrier body $\mathcal{B}_1$, and the centers of mass of rotors $\mathcal{D}_i$ are at the center of mass of the carrier body $\mathcal{B}_2$. It can be easily found from Figure 4.3.1 that we have the following kinematic constraints,

$$r_{S_i} = r_1, \qquad i = 1, 2, 3, \qquad (4.3.4a)$$

$$r_{D_i} = r_2, \qquad i = 1, 2, 3, \qquad (4.3.4b)$$

$$r_2 = r_1 - B_1 d_1 + B_2 d_2, \qquad (4.3.4c)$$

$$(m_1 + m_S) r_1 = -(m_2 + m_D) r_2, \qquad (4.3.4d)$$

with

$$m_S = m_{S_1} + m_{S_2} + m_{S_3},$$

$$m_D = m_{D_1} + m_{D_2} + m_{D_3},$$

where m_1, m_{S_i}, m_2, m_{D_i} are the masses of the corresponding bodies. Relation (4.3.4d) realize the fact that the reference inertial frame is located at the center of mass of this mechanical system. By using standard techniques, cf. (3.5.2), the total kinetic energy of the system can be written as

$$T = \frac{1}{2} m_1 |\dot{r}_1|^2 + \frac{1}{2} tr\, (\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} m_2 |\dot{r}_2|^2 + \frac{1}{2} tr\, (\dot{B}_2 I_2 \dot{B}_2^T)$$

$$+ \sum_{i=1}^{3} \left(\frac{1}{2} m_{S_i} |\dot{r}_{S_i}|^2 + \frac{1}{2} tr(\dot{S}_i I_{S_i} \dot{S}_i^T) \right)$$

$$+ \sum_{i=1}^{3} \left(\frac{1}{2} m_{D_i} |\dot{r}_{D_i}|^2 + \frac{1}{2} tr(\dot{D}_i I_{D_i} \dot{D}_i^T) \right),$$

where I_1, I_2, I_{S_i}, I_{D_i} are the coefficients of inertia of the corresponding bodies. Assuming there is no potential energy, the Lagrangian on the tangent bundle to the unconstrained configuration space Q_u can be written as, cf. (3.5.3),

$$\tilde{L} = \frac{1}{2} tr\, (\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} tr(\dot{B}_2 I_2 \dot{B}_2^T) + \frac{1}{2} \varepsilon\, |\dot{B}_1 d_1 - \dot{B}_2 d_2|^2$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \left(tr(\dot{S}_i I_{S_i} \dot{S}_i^T) + tr(\dot{D}_i I_{D_i} \dot{D}_i^T) \right),$$

$$(4.3.5)$$

where

$$\varepsilon \; \stackrel{\triangle}{=} \; \frac{(m_1 \; + \; m_S)(m_2 \; + \; m_D)}{m_1 \; + \; m_S \; + \; m_2 \; + \; m_D}$$

is the reduced mass. Now we let

$$\dot{B}_1 \; = \; B_1 \hat{\Omega}_1, \qquad \dot{B}_2 \; = \; B_2 \; \hat{\Omega}_2,$$

which are kinematics of $SO(3)$. Since $S_i \; = \; B_1 R(x_i, \theta_i)$, we have

$$\dot{S}_i = \; \dot{B}_1 R(x_i, \theta_i) \; + \; B_1 \dot{R}(x_i, \theta_i) \; = \; B_1(\hat{\Omega}_1 + \hat{s}_i) R(x_i, \theta_i).$$

and thus

$$\Omega_{S_i} \; = \; R(x_i, \theta_i)^T \Omega_1 + s_i,$$

where

$$s_1 \; = \; \begin{pmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 \; = \; \begin{pmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{pmatrix}, \quad s_3 \; = \; \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix}.$$

We can then write

$$tr(\dot{S}_i \, I_{S_i} \, \dot{S}_i^T) \; = \; tr\left((\hat{\Omega}_1 + \hat{s}_i) R(x_i, \theta_i) I_{S_i} R(x_i, \theta_i)^T (\hat{\Omega}_1 + \hat{s}_i)^T\right).$$

We naturally assume that the rotors have material symmetry about the axis of rotation, i.e.

$$R(x_i, \theta_i) \, I_{S_i} \, R(x_i, \theta_i)^T \; = \; I_{S_i},$$

and we get

$$tr(\dot{S}_i \, I_{S_i} \, \dot{S}_i^T) \; = \; tr\left((\hat{\Omega}_1 + \hat{s}_i) I_{S_i} (\hat{\Omega}_1 + \hat{s}_i)^T\right) \; = \; < \; \Omega_1 + s_i, \; I_{S_i}(\Omega_1 + s_i) \; >_E .$$

Similar derivations can be applied to the rotors $\mathcal{D}_i$. By substituting these formulae in (4.3.5), the Lagrangian, $\bar{L} \; : \; TQ \; \rightarrow \; \mathbb{R}$, can then be written as

69

$$\bar{L}(B_1, \ \theta_i, \ B_2, \ \phi_i, \ \Omega_1, \ \dot{\theta}_i, \ \Omega_2, \ \dot{\phi}_i, \ i = 1, 2, 3)$$

$$= \frac{1}{2} < \Omega_1, \ \mathbf{J}_1 \Omega_1 >_E + \frac{1}{2} < \Omega_2, \ \mathbf{J}_2 \Omega_2 >_E + \varepsilon < \Omega_1, \ \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2 >_E$$

$$+ \frac{1}{2} < \dot{\Theta}, \mathbf{I}^S \dot{\Theta} >_E + < \Omega_1, \ \mathbf{I}^S \dot{\Theta} >_E + \frac{1}{2} < \dot{\Phi}, \mathbf{I}^D \dot{\Phi} >_E + < \Omega_2, \ \mathbf{I}^D \dot{\Phi} >_E \ .$$

$$(4.3.6a)$$

with

$$\mathbf{J}_1 \ = \ \mathbf{I}_1 \ + \ \varepsilon \ \hat{d}_1^T \ \hat{d}_1 + \ \sum_{i=1}^{3} \mathbf{I}_{S_i},$$

$$\mathbf{J}_2 \ = \ \mathbf{I}_2 \ + \ \varepsilon \ \hat{d}_2^T \ \hat{d}_2 + \ \sum_{i=1}^{3} \mathbf{I}_{D_i},$$

$$\mathbf{I}^S \ = \ \text{diag}((\mathbf{I}_{S_1})_1 \ , (\mathbf{I}_{S_2})_2 \ , (\mathbf{I}_{S_3})_3), \qquad (4.3.6b)$$

$$\mathbf{I}^D \ = \ \text{diag}((\mathbf{I}_{D_1})_1 \ , (\mathbf{I}_{D_2})_2 \ , (\mathbf{I}_{D_3})_3),$$

$$\Theta \ = \ \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \qquad \Phi \ = \ \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

where $(\mathbf{I}_{S_i})_j$ denotes the j-th diagonal element in the moment of inertia matrix $\mathbf{I}_{S_i}$. The physical meaning of $\mathbf{J}_1$ is that it is the total moment of inertia of $\mathcal{B}_1$ plus rotors referred to the joint. The notation $<, >_E$ refers to the Euclidean inner product. For simplicity, we omit the subscript E in the following.

With this Lagrangian (4.3.6), we now apply a similar theorem as Theorem 2.2.1 to derive the dynamical equations for the multibody dual-spin problem. Let L be the Lagrangian function expressed in terms of the variables

$$(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi}) \ \in \ TQ.$$

First, we need to find the differential of L in a form analogous to (2.2.6). It can be found by the following procedure. Let $(U_1, U_2, U_3, U_4, W_1, W_2, W_3, W_4) \in T_{(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi})} TQ$, which can be written as the form, cf. (2.2.3),

$$(B_1 \ \hat{u}_1, \ u_2, \ B_2 \ \hat{u}_3, \ u_4, \ B_1(\hat{u}_1 \hat{\Omega}_1 + \hat{w}_1), \ w_2, \ B_2(\hat{u}_2 \hat{\Omega}_2 + \hat{w}_3), \ w_4).$$

It generates a curve in TQ given by

$$\left(B_1 e^{\epsilon \ \hat{u}_1}, \ \Theta + \epsilon u_2, \ B_2 e^{\epsilon \ \hat{u}_2}, \ \Phi + \epsilon u_4, \ B_1 e^{\epsilon \ \hat{u}_1}(\hat{\Omega}_1 + \epsilon \hat{w}_1), \ \dot{\Theta} + \epsilon w_2, \right.$$

$$\left. B_2 e^{\epsilon \ \hat{u}_2}(\hat{\Omega}_2 + \epsilon \hat{w}_3), \ \dot{\Phi} + \epsilon w_4 \right).$$

70

Here we have

$$dL \cdot (U_1, \ U_2, \ U_3, \ U_4, \ W_1, \ W_2, \ W_3, \ W_4)$$

$$= \frac{d}{d\epsilon} \Big|_{\epsilon=o} L\big(B_1 e^{\epsilon \, \hat{u}_1}, \ \Theta + \epsilon \, u_2, \ B_2 e^{\epsilon \, \hat{u}_2}, \ \Phi + \epsilon \, u_4,$$

$$B_1 e^{\epsilon \, \hat{u}_1}(\hat{\Omega}_1 + \epsilon \, \hat{w}_1), \ \dot{\Theta} + \epsilon \, w_2, \ B_2 e^{\epsilon \, \hat{u}_2}(\hat{\Omega}_2 + \epsilon \, \hat{w}_3), \ \dot{\Phi} + \epsilon \, w_4\big)$$

$$= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \bar{L}\big(B_1 e^{\epsilon \, \hat{u}_1}, \ \Theta + \epsilon \, u_2, \ B_2 e^{\epsilon \, \hat{u}_2}, \ \Phi + \epsilon \, u_4,$$

$$\Omega_1 + \epsilon \, w_1, \ \dot{\Theta} + \epsilon \, w_2, \ \Omega_2 + \epsilon \, w_3, \ \dot{\Phi} + \epsilon \, w_4\big)$$

$$= \ <D_{B_1}\bar{L}, B_1 \hat{u}_1> + \langle \frac{\partial \bar{L}}{\partial \Theta}, \ u_2 \rangle + <D_{B_2}\bar{L}, B_2 \hat{u}_3> + \langle \frac{\partial \bar{L}}{\partial \Phi}, \ u_4 \rangle$$

$$+ \langle \frac{\partial \bar{L}}{\partial \Omega_1}, \ w_1 \rangle + \langle \frac{\partial \bar{L}}{\partial \dot{\Theta}}, \ w_2 \rangle + \langle \frac{\partial \bar{L}}{\partial \Omega_2}, \ w_3 \rangle + \langle \frac{\partial \bar{L}}{\partial \dot{\Phi}}, \ w_4 \rangle.$$

The canonical form for $dL(B_1, \ \Theta, \ B_2, \ \Phi, \ \dot{B}_1, \ \dot{\Theta}, \dot{B}_2, \ \dot{\Phi})$ is, cf. (2.2.6),

$$\left(B_1(\hat{b}_1\hat{\Omega}_1 + \hat{a}_1), \ a_2, \ B_2(\hat{b}_3\hat{\Omega}_2 + \hat{a}_3), \ a_4, \ B_1\hat{b}_1, \ b_2, \ B_2\hat{b}_3, \ b_4 \right).$$

Let N_1, N_2 be given by the formula

$$D_{B_1}\bar{L} \ = \ B_1\hat{N}_1, \qquad D_{B_2}\bar{L} \ = \ B_2\hat{N}_3,$$

we have

$$a_1 \ = \ N_1, \quad a_2 \ = \ \frac{\partial \bar{L}}{\partial \Theta}, \quad a_3 \ = \ N_3, \quad a_4 \ = \ \frac{\partial \bar{L}}{\partial \Phi},$$

$$b_1 \ = \ \frac{\partial \bar{L}}{\partial \Omega_1}, \quad b_2 \ = \ \frac{\partial \bar{L}}{\partial \dot{\Theta}}, \quad b_3 \ = \ \frac{\partial \bar{L}}{\partial \Omega_2}, \quad b_4 \ = \ \frac{\partial \bar{L}}{\partial \dot{\Phi}},$$

and we get the form of elements in (2.2.7),

$$D_1 L \ = \ \left(B_1(\widehat{\frac{\partial \bar{L}}{\partial \Omega_1}}\hat{\Omega}_1 + \hat{N}_1), \ \frac{\partial \bar{L}}{\partial \Theta}, \ B_2(\widehat{\frac{\partial \bar{L}}{\partial \Omega_2}}\hat{\Omega}_2 + \hat{N}_3), \ \frac{\partial \bar{L}}{\partial \Phi} \right),$$

$$D_2 L \ = \ \left(B_1\widehat{\frac{\partial \bar{L}}{\partial \Omega_1}}, \ \frac{\partial \bar{L}}{\partial \dot{\Theta}}, \ B_2\widehat{\frac{\partial \bar{L}}{\partial \Omega_2}}, \ \frac{\partial \bar{L}}{\partial \dot{\Phi}} \right).$$

(4.3.7)

Next we assume there exist torques on each joint. We interpret these torques as horizontal 1-forms on TQ. In general, without the kinematic constraint, we have the general representation for a horizontal 1-form on the unconstrained tangent bundle, TQ_u,

$$\omega \ = \ B_1\hat{T}_1 \, dB_1 \ + \ \sum_{i=1}^{3} S_i\hat{T}_{S_i} \, dS_i \ + \ B_2\hat{T}_2 \, dB_2 \ + \ \sum_{i=1}^{3} D_i\hat{T}_{D_i} \, dD_i.$$

71

Assume that the torque on the spherical joint is T^J in the body 2 frame. T_{S_i}, T_{D_i} denote the torques exerted on the driven rotors and damping rotors respectively. We assume there is no external torques. By the nature of the torques, (Newton's third law), we know that

$$B_1 T_1 = -\sum_{i=1}^{3} S_i T_{S_i} - B_2 T^J$$

$$B_2 T_2 = -\sum_{i=1}^{3} D_i T_{D_i} + B_2 T^J$$

which implies

$$T_1 = -\sum_{i=1}^{3} R(x_i, \theta_i) T_{S_i} - B T^J$$

$$T_2 = -\sum_{i=1}^{3} R(x_i, \phi_i) T_{D_i} + T^J$$

where $B = B_1^T B_2$ is the *relative shape*. Let ω act on a special tangent vector

$$\omega \cdot (B_1 \hat{\Omega}_1, \dot{\Theta}, B_2 \hat{\Omega}_2, \dot{\Phi}, W_1, W_2, W_3, W_4)$$

$$= <T_1, \Omega_1> + \sum_{i=1}^{3} <T_{S_i}, R(x_i, \theta_i)^T \Omega_1 + s_i>$$

$$+ <T_2, \Omega_2> + \sum_{i=1}^{3} <T_{S_i}, R(x_i, \theta_i)^T \Omega_2 + d_i> \qquad (4.3.8)$$

$$= <-BT^J, \Omega_1> + \sum_{i=1}^{3} <T_{S_i}, s_i> + <T^J, \Omega_2> + \sum_{i=1}^{3} <T_{D_i}, d_i>$$

Thus the horizontal 1-form corresponding to the torques can be written as

$$\omega(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi})$$

$$= B_1(-\widehat{BT^J}) dB_1 + \sum_{i=1}^{3} (T_{S_i})_i d\theta_i + B_2 \hat{T}^J dB_2 + \sum_{i=1}^{3} (T_{D_i})_i d\phi_i$$

Let

$$T^S = \begin{pmatrix} (T_{S_1})_1 \\ (T_{S_2})_2 \\ (T_{S_3})_3 \end{pmatrix}, \qquad T^D = \begin{pmatrix} (T_{D_1})_1 \\ (T_{D_2})_2 \\ (T_{D_3})_3 \end{pmatrix},$$

where $(T_{S_i})_j$ denotes the j-th component of the vector T_{S_i}. Then

$$\alpha = \left(B_1(-\widehat{BT^J}), T^S, B_2 \widehat{T^J}, T^D \right).$$

We are ready to apply formula (2.2.7). The first component is worked out here. The others can be found in a similar way. From

$$B_1 \hat{\Omega}_1 \widehat{\frac{\partial \tilde{L}}{\partial \Omega_1}} + B_1 \frac{d}{dt} \widehat{\frac{\partial \tilde{L}}{\partial \Omega_1}} - B_1 \left(\widehat{\frac{\partial \tilde{L}}{\partial \Omega_1}} \hat{\Omega}_1 + \hat{N}_1 + (-\widehat{BT^J}) \right) = 0,$$

we get

$$\frac{d}{dt} M_1 = -\Omega_1 \times M_1 + N_1 - BT^J,$$

where $M_1 = \frac{\partial L}{\partial \Omega_1}$. This can be rewritten in terms of Ω_1, etc. Explicitly, we get the dynamical equations of the system in terms of variables in TQ.

$$\mathbf{J}_1 \dot{\Omega}_1 + \mathbf{I}^S \ddot{\Theta} + \varepsilon \, \hat{d}_1 B \hat{d}_2 \dot{\Omega}_2 = -\Omega_1 \times (\mathbf{J}_1 \Omega_1 + \mathbf{I}^S \dot{\Theta}) - \varepsilon \, \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2 - BT^J,$$

$$\mathbf{J}_2 \dot{\Omega}_2 + \mathbf{I}^D \ddot{\Phi} + \varepsilon \, \hat{d}_2 B^\tau \hat{d}_1 \dot{\Omega}_1 = -\Omega_2 \times (\mathbf{J}_2 \Omega_2 + \mathbf{I}^D \dot{\Phi}) - \varepsilon \, \hat{d}_2 B^\tau \hat{\Omega}_1 \hat{d}_1 \Omega_1 + T^J,$$

$$\mathbf{I}^S(\dot{\Omega}_1 + \ddot{\Theta}) = T^S,$$

$$\mathbf{I}^D(\dot{\Omega}_2 + \ddot{\Phi}) = T^D, \qquad\qquad (4.3.9)$$

$$\dot{B}_1 = B_1 \hat{\Omega}_1,$$

$$\dot{B}_2 = B_2 \hat{\Omega}_2.$$

It is easy to see that the RHS of the dynamical equations (4.3.9) only depend on the relative shape variable B. Thus we may do one immediate reduction, namely replacing the kinematics of B_1 and B_2 by the kinematics of B, or

$$\dot{B} = B \hat{\Omega}_2 - \hat{\Omega}_1 B. \qquad\qquad (4.3.10)$$

Thus the dimension of the dynamical system is dropped by 3. This corresponds to the Poisson reduction with respect to the $SO(3)$ action. The equilibrium points of the reduced system are termed *relative equilibria*.

For the realization of the multibody dual-spin control structure, the following *feedback laws* are used,

$$T^S = \mathbf{I}^S \dot{\Omega}_1,$$

$$T^D = -\beta \dot{\Phi}, \qquad\qquad (4.3.11)$$

$$T^J = -\gamma (\Omega_2 - B^\tau \Omega_1),$$

where β, and γ are positive definite matrices. The first equation in (4.3.11) makes the relative angular velocities between the driven rotors and $\mathcal{B}_1$ be constants. The other two equations signify damping torques on the damping rotors and the joint. With these feedback laws and (4.3.10), we may write the dynamical equations (4.3.9) as

$$\mathbf{J}_1\dot{\Omega}_1 + \varepsilon\,\hat{d}_1 B\hat{d}_2\dot{\Omega}_2 = -\Omega_1 \times (\mathbf{J}_1\Omega_1 + \mathbf{I}^S\dot{\Theta}) - \varepsilon\,\hat{d}_1 B\hat{\Omega}_2\hat{d}_2\Omega_2 + B\gamma(\Omega_2 - B^T\Omega_1),$$

$$\mathbf{J}_2\dot{\Omega}_2 + \varepsilon\,\hat{d}_2 B^T\hat{d}_1\dot{\Omega}_1 = -\Omega_2 \times (\mathbf{J}_2\Omega_2 + \mathbf{I}^D\dot{\Phi}) - \varepsilon\,\hat{d}_2 B^T\hat{\Omega}_1\hat{d}_1\Omega_1$$

$$- \gamma(\Omega_2 - B^T\Omega_1) + \beta\dot{\Phi},$$

$$\mathbf{I}^D(\dot{\Omega}_2 + \ddot{\Phi}) = -\beta\dot{\Phi},$$

$$\dot{B} = B\hat{\Omega}_2 - \hat{\Omega}_1 B,$$

$$(4.3.12)$$

where $\dot{\Theta}$ is a constant vector. Let

$$p_1 = \mathbf{J}_1\,\Omega_1 + \varepsilon\,\hat{d}_1\,B\,\hat{d}_2\,\Omega_2, \qquad (4.3.13a)$$

$$p_2 = \mathbf{I}^S\dot{\Theta}, \qquad (4.3.13b)$$

$$p_3 = (\mathbf{J}_2 - \mathbf{I}^D)\,\Omega_2 + \varepsilon\,\hat{d}_2\,B^T\,\hat{d}_1\,\Omega_1, \qquad (4.3.13c)$$

$$p_4 = \mathbf{I}^D(\Omega_2 + \dot{\Phi}), \qquad (4.3.13d)$$

be the corresponding conjugate momenta, we can express the dynamical equations (4.3.9) in terms of p_i variables,

$$\dot{p}_1 = -\tilde{\Omega}_1 \times (p_1 + p_2) - \varepsilon\,\widehat{\hat{d}_1\tilde{\Omega}_1}B\hat{d}_2\tilde{\Omega}_2 - \gamma(\tilde{\Omega}_1 - B\tilde{\Omega}_2),$$

$$\dot{p}_2 = 0.$$

$$\dot{p}_3 = -\tilde{\Omega}_2 \times (p_3 + p_4) - \varepsilon\,\widehat{\hat{d}_2\tilde{\Omega}_2}B^T\hat{d}_1\tilde{\Omega}_1 - \gamma(\tilde{\Omega}_2 - B^T\tilde{\Omega}_1) + \beta\dot{\tilde{\Phi}}, \qquad (4.3.14)$$

$$\dot{p}_4 = -\beta\dot{\tilde{\Phi}},$$

$$\dot{B} = B\hat{\tilde{\Omega}}_2 - \hat{\tilde{\Omega}}_1 B.$$

Here $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, and $\dot{\tilde{\Phi}}$ are the expressions of Ω_1, Ω_2, and $\dot{\Phi}$ in terms of p_i, respectively. These expressions can be found through (4.3.13). Later on equations (4.3.12) will be referred to as dynamics on the TQ side, while equations (4.3.14) will be referred to as dynamics

on the T^*Q side. By the symmetry of the system, we immediately have a first integral of the system, namely, the magnitude of total angular momentum vector is conserved,

$$|p_1 + p_2 + B(p_3 + p_4)|^2 = \text{constant.} \qquad (4.3.15)$$

4.4. Asymptotic Stability

The multibody dual-spin problem can be put into the general framework of gyroscopic systems with symmetry with exterior force. With a constant $\dot\Theta$, the configuration space should include only B_1, B_2, and Φ. Thus we may drop the quadratic term in $\dot\Theta$ in (4.3.6), and get the Lagrangian

$$\bar{L}(B_1, \ B_2, \ \phi_i, \ \Omega_1, \ \Omega_2, \ \dot\Phi)$$
$$= \frac{1}{2} <\Omega_1, \ \mathbf{J}_1\Omega_1> + \frac{1}{2} <\Omega_2, \ \mathbf{J}_2\Omega_2> + \varepsilon <\Omega_1, \ \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2> \qquad (4.4.1)$$
$$+ \frac{1}{2} <\dot\Phi, \mathbf{I}^D\dot\Phi> + <\Omega_2, \ \mathbf{I}^D\dot\Phi> + <\Omega_1, \ \mathbf{I}^S\dot\Theta> .$$

The last term in the Lagrangian is the so-called gyroscopic term, since it is linear in Ω_1. We are now ready to establish an asymptotic stability theorem. For this system, the torque formula (4.3.11) give rise to a dissipative force in the sense of Definition 4.1.2. In fact, from (4.3.8), we have

$$\omega(X_{H_L})(B_1, \ B_2, \ \dot\Phi, \dot{B}_1, \dot{B}_2, \dot\Phi)$$
$$= <-BT^J, \ \Omega_1> + <T^J, \ \Omega_2> + <T^D, \ \dot\Phi> \qquad (4.4.2)$$
$$= - <\dot\Phi, \ \beta\dot\Phi> - <\Omega_2 - B^T\Omega_1, \ \gamma(\Omega_2 - B^T\Omega_1)> .$$

It is readily checked that this exterior force satisfies the conditions in Definition 4.1.2, and thus is a dissipative force. As discussed in Section 4.2, the energy function for this system,

$$H_L = \frac{1}{2} <\Omega_1, \ \mathbf{J}_1\Omega_1> + \frac{1}{2} <\Omega_2, \ \mathbf{J}_2\Omega_2> + \varepsilon <\Omega_1, \ \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2>$$
$$+ \frac{1}{2} <\dot\Phi, \mathbf{I}^D\dot\Phi> + <\Omega_2, \ \mathbf{I}^D\dot\Phi>, \qquad (4.4.3)$$

is a suitable Lyapunov function. This is an analog of the *core energy* used in [34] to justify the energy-sink method and used in [36] to prove asymptotic stability. In the

absence of damping (i.e. $\gamma = \beta = 0$), the system is a hamiltonian system on the TQ side with Hamiltonian function H_L. This is clear from the construction.

We may now apply LaSalle Invariance Principle. Consider the dynamical equation (4.3.12). First, we define the *momentum variety* in TQ/G as,

$$M^{\mu}_{p_2} = \{(\Omega_1, \Omega_2, \dot{\Phi}, B) \in \mathbb{R}^9 \times SO(3) : |\, p_1 + p_2 + B(p_3 + p_4)\,|^2 = \mu^2\}, \qquad (4.4.4)$$

which is parametrized by p_2, and μ, since p_2 is a constant along any trajectory. Here p_1, p_3, p_4 should be thought as functions of the variables in the underlying space. From (4.3.15), the dynamical motion leaves the momentum variety invariant. This will be the domain of our analysis later on.

The set $\mathcal{R}$ discussed in Section 4.2 could be now written as

$$\mathcal{R} = \{\, (\Omega_1, \Omega_2, \dot{\Phi}, B) \in M^{\mu}_{p_2} \; : \; \dot{\Phi} = 0, \; \Omega_2 - B^T \Omega_1 = 0 \,\}. \qquad (4.4.5)$$

From (4.3.14), the set of equilibria in $M^{\mu}_{p_2}$ can be written as

$$
\begin{aligned}
\sum\nolimits_{\mu, p_2} = \Big\{ & (\Omega_1, \Omega_2, \dot{\Phi}, B) \in \mathbb{R}^9 \times SO(3) : \; |\, p + p_2\,|^2 = \mu^2, \\
& p_1 = p - (B\mathbf{J}_2 + \varepsilon\, \hat{d}_1\, B\, \hat{d}_2)(\mathbf{I}^D)^{-1} p_4, \\
& p_3 = \varepsilon\, \hat{d}_2 B^T \hat{d}_1 \mathbf{J}_1^{-1} p + (\mathbf{J}_2 - \mathbf{I}^D - \varepsilon\, \hat{d}_2 B^T \hat{d}_1 \mathbf{J}_1^{-1} B\mathbf{J}_2)(\mathbf{I}^D)^{-1} p_4, \\
& \dot{\Phi} = 0, \quad \Omega_2 - B^T \Omega_1 = 0, \\
& -\Omega_1 \times (\mathbf{J}_1 \Omega_1 + \mathbf{I}^S \dot{\Theta}) - \varepsilon\, \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2 = 0, \\
& -\Omega_2 \times \mathbf{J}_2 \Omega_2 - \varepsilon\, \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1 = 0 \Big\}
\end{aligned}
$$

This set is in one-to-one correspondence with the set of equilibria of the gyroscopic system obtained by setting $\dot{\Phi} = 0$. See also Section 7.3. Now we characterize the maximal invariant set in $\mathcal{R}$. Since $\mathcal{R}$ can be written as the level set of two functions,

$$f_1(\Omega_1, \Omega_2, \dot{\Phi}, B) = \dot{\Phi},$$

$$f_2(\Omega_1, \Omega_2, \dot{\Phi}, B) = \Omega_2 - B^T \Omega_1,$$

we can find the maximal invariant set using (4.2.7) and these two functions. For $k = 1$ in (4.2.7), we ask,

$$L_X f_1 = 0, \quad L_X f_2 = 0, \quad \text{on } \mathcal{R},$$

76

which, in term, give us the conditions

$$\dot{\Omega}_1 = 0, \qquad \dot{\Omega}_2 = 0.$$

From (4.3.12), we then get

$$-\Omega_1 \times (J_1\Omega_1 + I^S\dot{\Theta}) - \varepsilon \hat{d}_1 B\hat{\Omega}_2 \hat{d}_2 \Omega_2 = 0,$$

$$-\Omega_2 \times J_2\Omega_2 - \varepsilon \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1 = 0.$$

We thus proved that the maximal invariant set is exactly the same as the set $\sum_{\mu,p_2}$ of equilibria for (4.3.12). From Theorem 4.2.3, we know that each trajectory will approach the maximal invariant set in the limit. We can thus conclude the following theorem.

THEOREM 4.4.1

The mechanical system (4.3.12) asymptotically approaches one of the stable equilibria in $\sum_{\mu,p_2}$, or the equilibria of the limiting hamiltonian system.

∎

The limiting motions of the system, i.e. the relative equilibria, could be characterized by the Principle of Symmetric Criticality with gyroscopic term. The system behavior is affected by changing the gyroscopic field, which corresponds to altering the driven rotor velocity here. These issues will be discussed in more detail later.

4.5. Partial Damping

For coupled hamiltonian systems with added dissipation, it is not necessary to put damping mechanism at every interconnection. This is because, intuitively, the energy will transmit from one body to the other bodies, and thus will be damped out at some connections by the damping mechanism there. On the other hand, if we only put one damping unit, the system may be driven into a nontrivial motion. The following two examples discuss these cases clearly.

EXAMPLE 4.5.1

We consider the mechanical system described in Figure 4.5.1. This is a simple system including four bodies lined up and connected with springs. In this example, we put the dashpot (damping unit) β in the last spring.
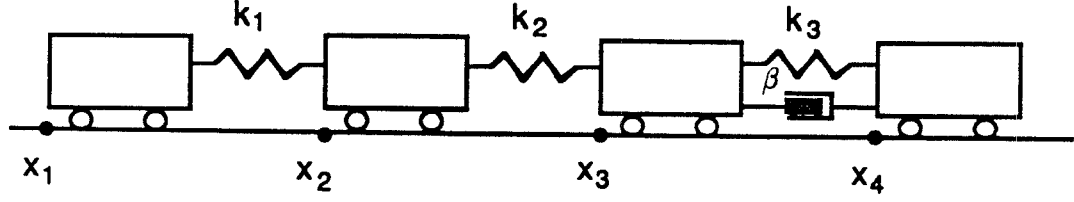
Figure 4.5.1. Four Bodies Connected by Springs I

The configuration space for this system is $\mathbb{R}^4$, denoted by $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Assuming the reference lengths of the springs are a_1, a_2, a_3 respectively. We could write the kinetic energy and potential energy as follows.

$$T = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2 + m_4 \dot{x}_4^2\right),$$

$$V = \frac{1}{2} k_1 (x_2 - x_1 - a_1)^2 + \frac{1}{2} k_2 (x_3 - x_2 - a_2)^2 + \frac{1}{2} k_3 (x_4 - x_3 - a_3)^2.$$

(4.5.1)

The Lagrangian for this system is $L = T - V$. The exterior force is a viscous friction coming from the dashpot. We may model it as a horizontal 1-form in the following way

$$\omega = (0,\ 0,\ -\beta(\dot{x}_3 - \dot{x}_4),\ -\beta(\dot{x}_4 - \dot{x}_3),\ 0,\ 0,\ 0,\ 0). \tag{4.5.2}$$

We now apply Lagrange-d'Alembert Principle. The dynamical equations of this system could be found as

$$m_1 \ddot{x}_1 = k_1(x_2 - x_1 - a_1),$$

$$m_2 \ddot{x}_2 = -k_1(x_2 - x_1 - a_1) + k_2(x_3 - x_2 - a_2),$$

$$m_3 \ddot{x}_3 = -k_2(x_3 - x_2 - a_2) + k_3(x_4 - x_3 - a_3) - \beta(\dot{x}_3 - \dot{x}_4),$$

$$m_4 \ddot{x}_4 = -k_3(x_4 - x_3 - a_3) - \beta(\dot{x}_4 - \dot{x}_3).$$

(4.5.3)

It is obvious from equation (4.5.3), also from the Lagrangian (4.5.1), that the system is invariant under a translational motion. We may thus do one reduction with respect to this symmetry. Let $\mathbf{v} = (v_1, v_2, v_3, v_4) = \dot{\mathbf{x}}$, and

$$d_1 = x_2 - x_1, \quad d_2 = x_3 - x_2, \quad d_3 = x_4 - x_3.$$

The reduced dynamics may be written as, from (4.5.3),

78

$$\dot{d}_1 = v_2 - v_1, \tag{4.5.4a}$$

$$\dot{d}_2 = v_3 - v_2, \tag{4.5.4b}$$

$$\dot{d}_3 = v_4 - v_3, \tag{4.5.4c}$$

$$m_1\dot{v}_1 = k_1(d_1 - a_1), \tag{4.5.4d}$$

$$m_2\dot{v}_2 = -k_1(d_1 - a_1) + k_2(d_2 - a_2), \tag{4.5.4e}$$

$$m_3\dot{v}_3 = -k_2(d_2 - a_2) + k_3(d_3 - a_3) - \beta(v_3 - v_4), \tag{4.5.4f}$$

$$m_4\dot{v}_4 = -k_3(d_3 - a_3) - \beta(v_4 - v_3). \tag{4.5.4g}$$

Relative equilibria are those states satisfying

$$d_1 = a_1, \quad d_2 = a_2, \quad d_3 = a_3,$$
$$v_1 = v_2 = v_3 = v_4 = \text{constant}, \tag{4.5.5}$$

namely, the four bodies move at the same speed without deformations in the springs. It is easy to see that ω in (4.5.2) is a dissipative force and the solutions approach the maximal invariant set in

$$\mathcal{R} = \{ (d_1, d_2, d_3, \mathbf{v}) : v_3 - v_4 = 0 \}.$$

The dynamical equations (4.5.4) restricted to $\mathcal{R}$ can be found as, with $d_3 = d_{30}$ a constant,

$$\dot{d}_1 = v_2 - v_1, \tag{4.5.6a}$$

$$\dot{d}_2 = v_3 - v_2, \tag{4.5.6b}$$

$$m_1\dot{v}_1 = k_1(d_1 - a_1), \tag{4.5.6e}$$

$$m_2\dot{v}_2 = -k_1(d_1 - a_1) + k_2(d_2 - a_2), \tag{4.5.6d}$$

$$m_3\dot{v}_3 = -k_2(d_2 - a_2) + k_3(d_3 - a_3), \tag{4.5.6e}$$

$$m_4\dot{v}_4 = -k_3(d_3 - a_3). \tag{4.5.6f}$$

We are looking for the initial conditions of (4.5.6) satisfying

$$d_3(0) = d_{30}, \quad v_3(0) = v_4(0),$$
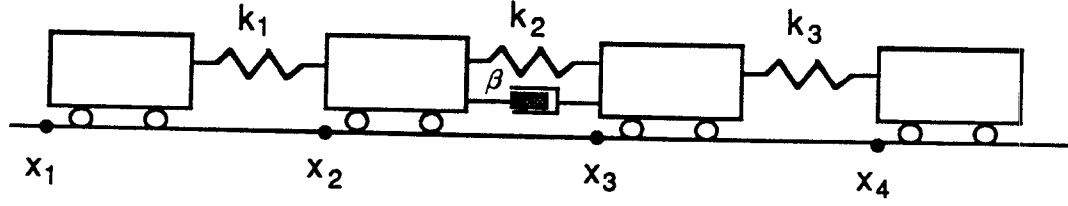
Figure 4.5.2. Four Bodies Connected by Springs II

and leave the trajectories stay in $\mathcal{R}$. We require that $\dot{v}_3 = \dot{v}_4$ and get the relation, from (4.5.6d), (4.5.6e),

$$m_4 k_2 (d_2 - a_2) = (m_3 + m_4) k_3 (d_{30} - a_3).$$

Since all the elements except d_2 are constant, d_2 must be a constant as well. Thus from (4.5.6b), we have $v_2 = v_3$. By similar arguments, we get the following conditions for the states along the trajectories,

$$d_1 = d_{10}, \quad d_2 = d_{20}, \quad d_3 = d_{30},$$

$$v_1 = v_2 = v_3 = v_4 = v.$$

On the other hand, there is a conserved quantity of this system, i.e. the total linear momentum,

$$m_1 v_1 + m_2 v_2 + m_3 v_3 + m_4 v_4 = \text{const.}$$

It follows that $v = v_0$, a constant speed, and

$$d_{10} = a_1, \quad d_{20} = a_2, \quad d_{30} = a_3.$$

Consequently, only the initial conditions satisfying (4.5.5) will leave the restricted dynamics stay in $\mathcal{R}$. The maximal invariant set in $\mathcal{R}$ is thus the same as the set of relative equilibria. We conclude the asymptotic stability of this system.

∎

EXAMPLE 4.5.2

We consider a mechanical system similar to the one discussed in Example 4.5.1. The only difference is that we put the dashpot (damping unit) β in the middle spring, as depicted in Figure 4.5.2.

80

The Lagrangian for this system is the same as before, while the exterior force is different now, cf.(4.5.2),

$$\omega = (0, \ -\beta(\dot{x}_2 - \dot{x}_3), \ -\beta(\dot{x}_3 - \dot{x}_2), \ 0, \ 0, \ 0, \ 0, \ 0). \tag{4.5.7}$$

Following similar arguments and notations, we get the reduced dynamical equations for this system, (from (4.5.1),(4.5.7)),

$$\dot{d}_1 = v_2 - v_1, \tag{4.5.8a}$$

$$\dot{d}_2 = v_3 - v_2, \tag{4.5.8b}$$

$$\dot{d}_3 = v_4 - v_3, \tag{4.5.8c}$$

$$m_1 \dot{v}_1 = k_1(d_1 - a_1), \tag{4.5.8d}$$

$$m_2 \dot{v}_2 = -k_1(d_1 - a_1) + k_2(d_2 - a_2) - \beta(v_2 - v_3), \tag{4.5.8e}$$

$$m_3 \dot{v}_3 = -k_2(d_2 - a_2) + k_3(d_3 - a_3) - \beta(v_3 - v_2), \tag{4.5.8f}$$

$$m_4 \dot{v}_4 = -k_3(d_3 - a_3). \tag{4.5.8g}$$

The conditions for relative equilibria are the same as before, i.e. (4.5.5). We are looking for the maximal invariant set in

$$\mathcal{R} = \{ \ (d_1, d_2, d_3, \mathbf{v}) \ : \ v_2 - v_3 = 0 \ \}.$$

The dynamics restricted to $\mathcal{R}$ could be written as, with $d_2 = d_{20}$ a constant here, cf.(4.5.6),

$$\dot{d}_1 = v_2 - v_1, \tag{4.5.9a}$$

$$\dot{d}_3 = v_4 - v_3, \tag{4.5.9b}$$

$$m_1 \dot{v}_1 = k_1(d_1 - a_1), \tag{4.5.9e}$$

$$m_2 \dot{v}_2 = -k_1(d_1 - a_1) + k_2(d_2 - a_2), \tag{4.5.9d}$$

$$m_3 \dot{v}_3 = -k_2(d_2 - a_2) + k_3(d_3 - a_3), \tag{4.5.9e}$$

$$m_4 \dot{v}_4 = -k_3(d_3 - a_3), \tag{4.5.9f}$$

with initial conditions satisfying $v_2(0) = v_3(0)$. From the relation $\dot{v}_2 = \dot{v}_3$ in $\mathcal{R}$ and (4.5.9d), (4.5.9e), we get the equation

$$m_3 k_1 (d_1 - a_1) + m_2 k_3 (d_3 - a_3) \;=\; (m_2 + m_3) k_2 (d_{20} - a_2). \qquad (4.5.10)$$

By taking time derivatives on both side of (4.5.10), we obtain

$$m_3 k_1 \dot{d}_1 + m_2 k_3 \dot{d}_3 \;=\; 0.$$

With (4.5.9a), (4.5.9b), we find another relation,

$$-m_3 k_1 v_1 + m_2 k_3 v_4 + (m_3 k_1 - m_2 k_3) v_2 \;=\; 0.$$

Taking time derivative on each term once more and using (4.5.9c),(4.5.9d), (4.5.9f), we have

$$\left(-\frac{m_3 k_1^2}{m_1} - \frac{(m_3 k_1 - m_2 k_3) k_1}{m_2} \right) (d_1 - a_1) - \frac{m_2 k_3^2}{m_4} (d_3 - a_3)$$
$$= \frac{(m_3 k_1 - m_2 k_3) k_2}{m_2} (d_{20} - a_2). \qquad (4.5.11)$$

Now equations (4.5.10), (4.5.11) form two equations for two variables d_1, d_3. If these two equations are nonsingular, we could solve for d_1, d_3 which are then constants along the solution. It could be found that the condition for singularity is

$$\frac{m_2 k_3}{m_4}(m_3 - m_4) \;+\; \frac{m_4 k_1}{m_1}(m_2 - m_1) \;=\; 0. \qquad (4.5.12)$$

With similar arguments as the discussion at the end of Example 4.5.1, we conclude that generically the trajectories of the system (4.5.8) approach the set of relative equilibria asymptotically. For the system satisfying (4.5.12) the limiting dynamics exhibits nontrivial behavior. $\blacksquare$

The above two examples illustrate the point that partial damping may either ensure asymptotic stability or else drive the system into a resonance state. Now we consider the mechanical system discussed in Sections 4.3, 4.4 again. Instead of putting damping on both the ball-in-socket joint and damping rotors, we only install damping mechanism on the damping rotors. Namely, the torque law is now, cf. (4.3.11),

$$T^D = -\beta \dot{\Phi},$$
$$T^J = 0. \qquad (4.5.13)$$

The force generated by these torques is still a dissipative force, with the zero set, cf. (4.4.5),

$$\mathcal{R} = \{ (\Omega_1, \Omega_2, \dot{\Phi}, B) \in M_{p_2}^{\mu} : \dot{\Phi} = 0 \}. \tag{4.5.14}$$

The dynamics restricted to $\mathcal{R}$ could be written as, from (4.3.12),

$$J_1 \dot{\Omega}_1 + \varepsilon \hat{d}_1 B \hat{d}_2 \dot{\Omega}_2 = -\Omega_1 \times (J_1 \Omega_1 + I^S \dot{\Theta}) - \varepsilon \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2, \tag{4.5.15a}$$

$$J_2 \dot{\Omega}_2 + \varepsilon \hat{d}_2 B^T \hat{d}_1 \dot{\Omega}_1 = -\Omega_2 \times J_2 \Omega_2 - \varepsilon \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1, \tag{4.5.15b}$$

$$I^D(\dot{\Omega}_2 + \ddot{\Phi}) = 0, \tag{4.5.15c}$$

$$\dot{B} = B \hat{\Omega}_2 - \hat{\Omega}_1 B, \tag{4.5.15d}$$

with initial conditions satisfying $\dot{\Phi}(0) = 0$. To have the solutions in $\mathcal{R}$, it is required that

$$\ddot{\Phi} = 0.$$

It follows that, from (4.5.15c), that $\Omega_2 = \Omega_{20}$, a constant vector. Thus (4.5.15) could be further written as

$$J_1 \dot{\Omega}_1 = -\Omega_1 \times (J_1 \Omega_1 + I^S \dot{\Theta}) - \varepsilon \hat{d}_1 B \hat{\Omega}_{20} \hat{d}_2 \Omega_{20}, \tag{4.5.16a}$$

$$\varepsilon \hat{d}_2 B^T \hat{d}_1 \dot{\Omega}_1 = -\Omega_{20} \times J_2 \Omega_{20} - \varepsilon \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1, \tag{4.5.16b}$$

$$\dot{B} = B \hat{\Omega}_{20} - \hat{\Omega}_1 B, \tag{4.5.16c}$$

We are looking for initial conditions such that the solution of (4.5.16) will stay in $\mathcal{R}$ forever. From (4.5.16a), (4.5.16b), we immediately have the relation

$$\hat{d}_2 B^T \hat{d}_1 J_1^{-1} \left(\Omega_1 \times (J_1 \Omega_1 + I^S \dot{\Theta}) + \varepsilon \hat{d}_1 B \hat{\Omega}_{20} \hat{d}_2 \Omega_{20} \right) - \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1$$
$$= \frac{1}{\varepsilon} \Omega_{20} \times J_2 \Omega_{20}. \tag{4.5.17}$$

For simplicity, we let

$$l \triangleq I^S \dot{\Theta},$$

$$\mathbf{P} \triangleq \Omega_1 \times (J_1 \Omega_1 + l) + \varepsilon \hat{d}_1 B \hat{\Omega}_{20} \hat{d}_2 \Omega_{20}.$$

83

By taking derivatives on each element in (4.5.17), we get

$$\hat{d}_2 B^T \left((\hat{\Omega}_1 - \widehat{B\Omega_{20}})(\hat{d}_1 \mathbf{J}_1^{-1} \mathbf{P} - \hat{\Omega}_1 \hat{d}_1 \Omega_1) + \varepsilon \, \hat{d}_1 \mathbf{J}_1^{-1} \hat{d}_1 B (\hat{\Omega}_{20} - \widehat{B^T \Omega_1}) \hat{\Omega}_{20} \hat{d}_2 \Omega_{20} \right.$$

$$\left. - \hat{d}_1 \mathbf{J}_1^{-1} \widehat{\mathbf{J}_1^{-1} \mathbf{P}} (\mathbf{J}_1 \Omega_1 + l) - \hat{d}_1 \mathbf{J}_1^{-1} \hat{\Omega}_1 \mathbf{P} + \widehat{\mathbf{J}_1^{-1} \mathbf{P}} \hat{d}_1 \Omega_1 + \hat{\Omega}_1 \hat{d}_1 \mathbf{J}_1^{-1} \mathbf{P} \right) = 0.$$

$$(4.5.18)$$

Equations (4.5.17), (4.5.18) form a set of six equations with six unknowns. If we could solve the equations for Ω_1, B, Theorem 4.4.1 could be established as well for the partial damping case.

4.6. Decoupling by Driven Torque Feedback

For coupled mechanical systems, such as the one shown in Figure 4.3.1, decoupling of motions is always an interesting issue. Here we consider the two bodies system and propose a scheme which makes the motion of one body be decoupled from the motion of the other body. The configuration is similar to the one discussed in Section 4.3, except that the joint damping mechanism and damping rotors are taken out, and the speed of driven rotors on body 1 is varied. From (4.3.9), the dynamical equations for this system can be written as,

$$\mathbf{J}_1 \dot{\Omega}_1 + \mathbf{I}^S \ddot{\Theta} + \varepsilon \, \hat{d}_1 B \hat{d}_2 \dot{\Omega}_2 = -\Omega_1 \times (\mathbf{J}_1 \Omega_1 + \mathbf{I}^S \dot{\Theta}) - \varepsilon \, \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2, \quad (4.6.1a)$$

$$\mathbf{J}_2 \dot{\Omega}_2 + \varepsilon \, \hat{d}_2 B^T \hat{d}_1 \dot{\Omega}_1 = -\Omega_2 \times \mathbf{J}_2 \Omega_2 - \varepsilon \, \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1, \quad (4.6.1b)$$

$$\mathbf{I}^S (\dot{\Omega}_1 + \ddot{\Theta}) = T^S, \quad (4.6.1c)$$

$$\dot{B}_1 = B_1 \hat{\Omega}_1, \quad (4.6.1d)$$

$$\dot{B}_2 = B_2 \hat{\Omega}_2. \quad (4.6.1e)$$

The control here is the driven torques T^S. It can be checked that with the control law,

$$T^S = -(\Omega_1 \times \mathbf{I}^S (\Omega_1 + \dot{\Theta}) + \varepsilon \, \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2 + \varepsilon \, \hat{d}_1 B \hat{d}_2 \dot{\Omega}_2), \quad (4.6.2)$$

equation (4.6.1a) can be found to be

$$(\mathbf{J}_1 - \mathbf{I}^S)\dot{\Omega}_1 \;=\; -\Omega_1 \times (\mathbf{J}_1 - \mathbf{I}^S)\Omega_1, \tag{4.6.3}$$

which is the Euler's equation for a rigid body. Thus the motion of Body 1 exhibits the single rigid body motion with modified inertia tensor. From (4.6.2) and (4.6.1b), we could further write the motion of Body 2 as

$$
\begin{aligned}
\mathbf{J}_2\dot{\Omega}_2 \;=\; & -\,\Omega_2 \times \mathbf{J}_2\Omega_2 - \varepsilon\,\hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1 \\
& +\, \varepsilon\,\hat{d}_2 B^T \hat{d}_1 (\mathbf{J}_1 - \mathbf{I}^S)^{-1}\big(\Omega_1 \times (\mathbf{J}_1 - \mathbf{I}^S)\Omega_1\big).
\end{aligned} \tag{4.6.4}
$$

By substituting the expression of $\dot{\Omega}_2$ in (4.6.2), we get a quadratic feedback law for the driven torques,

$$
\begin{aligned}
T^S \;=\; -\,\Big(& \Omega_1 \times \mathbf{I}^S(\Omega_1 + \dot{\Theta}) + \varepsilon\,\hat{d}_1 B\hat{\Omega}_2 \hat{d}_2 \Omega_2 + \varepsilon\,\hat{d}_1 B\hat{d}_2 \mathbf{J}_2^{-1}\Big[-\Omega_2 \times \mathbf{J}_2\Omega_2 \\
& -\, \varepsilon\,\hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1 + \varepsilon\,\hat{d}_2 B^T \hat{d}_1 (\mathbf{J}_1 - \mathbf{I}^S)^{-1}\big(\Omega_1 \times (\mathbf{J}_1 - \mathbf{I}^S)\Omega_1\big)\Big]\Big).
\end{aligned} \tag{4.6.5}
$$

With this velocity-dependent feedback (4.6.5), we could thus decouple the motion of Body 1 from the motion of Body 2, though the motion of Body 2 (4.6.4) is highly affected by the motion of Body 1.

CHAPTER V

Stability Analysis

In this chapter, we discuss the notion of relative stability and consider several schemes for determining stability. They are energy-Casimir method, Lagrange-multiplier method, and energy-momentum method. In particular, the block-diagonalization techniques associated to the energy-momentum method in determining stability for simple mechanical systems with symmetry are successfully extended to gyroscopic systems with symmetry. These techniques will be applied to specific problems in the following chapters.

5.1. Relative Stability

Let B, P be differentiable manifolds, and G be a Lie group. Consider a *principal G-bundle*, (P, G, B), namely, G acts differentiably on P freely and properly, $B = P/G$ is the quotient space of P with the canonical projection $\pi : P \to B$ being differentiable. Moreover, P is locally trivial, that is, every point $u \in B$ has a neighborhood U such that there is a mapping from $\pi^{-1}(U)$ to $U \times G$, $z \mapsto (\pi(z), \phi(z))$ which is a diffeomorphism and $\phi(g \cdot z) = g \cdot \phi(z)$, for all $g \in G$. See Figure 5.1.1 for an illustration of the geometric structure of such an object. For more details, see, e.g. [49].

A vector field X on P is said to be *projectable* if for each $\tilde{f} \in \mathcal{F}(B)$, there exists a $\bar{f} \in \mathcal{F}(B)$ such that

$$X[\, \tilde{f} \circ \pi \,] = \bar{f} \circ \pi,$$

where the LHS denotes the Lie derivative, cf, e.g. [46], [30] Now, given a projectable vector field X on P, the corresponding *projected vector field* $\tilde{X}$ on B is defined in the following way. Let $\tilde{f}$ be a smooth function on B, the Lie derivative of $\tilde{X}$ on $\tilde{f}$ is defined through
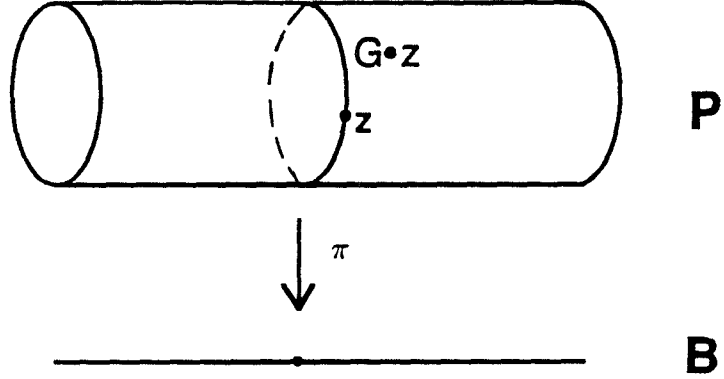
Figure 5.1.1. Principal G-bundle

$$\tilde{X}[\,\tilde{f}\,] \,\triangleq\, \tilde{f}, \quad \text{or} \quad \tilde{X}[\,\tilde{f}\,]\circ\pi \,=\, X[\,\tilde{f}\circ\pi\,]. \tag{5.1.1}$$

It is easy to verify that the vector field X_h defined in (3.3.2) is projectable with the projected vector field $X_{\bar{h}}$ defined in (3.3.3) in the above sense. Now, we have the following notion.

DEFINITION 5.1.1

For the principal G-bundle, (P, G, B), a point $z \in P$ is called a *relative equilibrium* of a projectable vector field $X \in \mathfrak{X}(P)$ if $\pi(z)$ is an equilibrium of the associated projected vector field $\bar{X} \in \mathfrak{X}(B)$. Moreover, a relative equilibrium $z \in P$ is *relatively stable modulo G* if the equilibrium $\pi(z)$ is Lyapunov stable with respect to the projected vector field $\bar{X}$.

∎

REMARK 5.1.2

In [43], the smooth manifold structure of the quotient space P/G is not explicitly invoked in defining the notion of *stationary motion* and *relative stability modulo G*. However, for the group action being free and proper, or P/G is a manifold, which is the case considered in this dissertation, Definition 8.13, p. 242 in [43] are equivalent to Definition 5.1.1.

∎

For a gyroscopic system with symmetry, the definition of relative equilibrium

$v_x \in TQ$ in Definition 3.4.2 matches with the Definition 5.1.1 by noting that the principal G-bundle is now $(TQ, G, TQ/G)$. Accordingly, the relative equilibrium v_x is *relatively stable modulo G in TQ* if $\tilde{\tau}(v_x)$ is a stable equilibrium with respect to the projected Hamiltonian vector field $X_{\tilde{H}_L}$. On the other hand, in the symplectic reduction process, we have the bundle structure $(J^{-1}(\mu),\ G_\mu,\ (TQ)_\mu)$. The relative equilibrium defined in Definition 3.4.1 can be regarded as a relative equilibrium with respect to this principal G-bundle. Correspondingly, we may define the *relative stability modulo G_μ in $J^{-1}(\mu)$* with respect to the reduced dynamics $X_{H_L^\mu}$. Since the space $(TQ)_\mu$ is diffeomorphic to a symplectic leaf in TQ/G, relative stability modulo G in TQ implies relative stability modulo G_μ in $J^{-1}(\mu)$. The converse is illustrated by the following theorem from [43], Theorem 8.17, p. 244, see also [37].

THEOREM 5.1.3

Let v_x^e be a relative equilibrium, cf. Definition 3.4.1 or 3.4.2. Definiteness of the Hessian $D^2 H_L^\mu$ at $\pi_\mu(v_x^e) \in (TQ)_\mu$ implies the relative stability modulo G in TQ of v_x^e if there exists a neighborhood W of $\tilde{\tau}(v_x^e) \in TQ/G$ such that the rank of the Poisson structure $\{\cdot, \cdot\}_i$, defined in (3.3.5), is constant in W.

∎

Those points v_x in TQ satisfying the constant-rank condition stated in the above theorem will be referred to as *generic points*. The following example demonstrates that the sufficient condition in Theorem 5.1.3 is essential. This example is from [43]. A detailed discussion can be also found in [37].

EXAMPLE 5.1.4

Consider a symplectic manifold $(Q,\ \omega)$, where

$$Q = \mathbb{R}^4 = \{\ (q_1,\ q_2,\ p_1,\ p_2)\ \},$$

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2. \tag{5.1.2}$$

Let

$$G = Aff_+(\mathbb{R})$$

$$\stackrel{\triangle}{=} \{\ (a, b) \in \mathbb{R}^2\ \} \text{ with the group structure} \tag{5.1.3}$$

$$(a, b) \cdot (c, d) = (a + c,\ b + e^a d).$$

It can be shown that G defined in (5.1.3) is a Lie group. We define the action of G on Q as

$$G \times Q \rightarrow Q,$$

$$((a,b), (q_1,q_2,p_1,p_2)) \mapsto (a + q_1, \ b + e^a q_2, \ p_1, \ e^{-a} p_2).$$

It is easy to check that this is a symplectic action on Q. This action is also free and proper. It follows that Q/G is a manifold ($\simeq \mathbb{R}^2$). The symplectic structure ω in (5.1.2) defines a Poisson bracket on $\mathcal{F}(Q)$ which, in turn, induces a Poisson structure on Q/G. Let a Hamiltonian function H be defined as

$$H(q_1,q_2,p_1,p_2) \ = \ p_2 \ e^{q_1},$$

which is a G-invariant function. It can be checked that $(0,t,0,0) \in Q$ is a relative equilibrium corresponding to the vector field X_H. Moreover, this relative equilibrium is relatively stable modulo G_μ in $J^{-1}(\mu)$, since the quotient space $J^{-1}(\mu)/G_\mu$ degenerates to a point. However, it has been shown in [37] that this relative equilibrium is not relatively stable modulo G in Q. Note also that that the induced Poisson structure doesn't have a constant rank at $(0,t,0,0)$ and hence the condition in Theorem 5.1.3 does not hold.

∎

In the remaining sections of this Chapter, several useful methods will be discussed for determining relative stability in the appropriate sense. For simplicity, we will drop the underlying space in the definition of relative stability, e.g. we say merely *relative stability modulo G_μ*. The underlying space is clear from the construction.

5.2. Energy-Casimir Method

We first recall some facts from the Lyapunov's stability analysis. Consider the dynamical system

$$\dot{x} \ = \ f(x), \tag{5.2.1}$$

where $f : W \rightarrow \mathbb{R}^n$ is a C^1 map on an open set W in $\mathbb{R}^n$. We have the following lemma.

<u>LEMMA 5.2.1</u>   (See [32], p. 193)

Let $\bar{x} \in W$ be an equilibrium of the system (5.2.1), $f(\bar{x}) = 0$. Let $V : U \to \mathbb{R}$ be a continuous function defined on U, a neighborhood of $\bar{x}$ in W, differentiable on $U - \bar{x}$, such that

$$V(\bar{x}) \ = \ 0,$$

$$V(x) \ > \ 0, \quad \forall x \in U - \bar{x},$$

$$\dot{V}(x) \ \leq \ 0, \quad \forall x \in U - \bar{x},$$

then $\bar{x}$ is Lyapunov stable. Here $\dot{V}(x) = DV(x) \cdot f(x)$ is the directional derivative of the function V along the trajectory.

∎

Now we consider a general Poisson system with a Hamiltonian H,

$$\dot{x} \ = \ \{x,\ H\}(x) \ = \ \Lambda(x)\ \nabla H(x). \tag{5.2.2}$$

A function f on a Poisson manifold $(P,\ \{\cdot,\cdot\}_P)$ is said to be a *Casimir function* if

$$\{\ f,\ \psi\ \}_P \ = \ 0 \qquad \forall\ \psi \in C^\infty(P).$$

A Casimir function is automatically a conserved quantity for *any* Hamiltonian vector field X_H on P. This can be seen from the following identity,

$$L_{X_H} f \ = \ \{f,\ H\}_P.$$

If $\{\cdot,\cdot\}_P$ is induced from a symplectic structure and P is connected, then the only Casimir functions are the constant functions.

For the system (5.2.2), assume the null space of Λ is not empty and spanned by $\nabla C_i, i = 1, \cdots, m$, where C_i are Casimir functions. Then x_e is an equilibrium of (5.2.2) if and only if

$$\nabla H(x_e) \ = \ \sum_{i=1}^{m} \lambda_i\ \nabla C_i(x_e), \tag{5.2.3}$$

or

$$\nabla \left(H - \sum_{i=1}^{m} \lambda_i\ C_i \right)(x_e) \ = \ 0.$$

Since C_i are Casimirs, we may write (5.2.2) as

$$\dot{x} = \{x, \, H - \sum_{i=1}^{m} \lambda_i \, C_i\}(x).$$

It follows that $H - \sum_{i=1}^{m} \lambda_i \, C_i$ is a conserved quantity along the trajectories of (5.2.2). Notice that if C is a Casimir, so is any smooth functional of C. We have the following theorem.

THEOREM 5.2.2   (Energy-Casimir)

   If there exists a Casimir function C such that

$$\nabla(H + C)(x_e) \; = \; 0, \tag{5.2.4a}$$

$$(\text{second variation}) \quad \nabla^2(H + C)(x_e) \; > \; 0, \quad (\text{or} \; < 0) \tag{5.2.4b}$$

then x_e is a Lyapunov stable equilibrium of (5.2.2).

Proof

   Define

$$V(x) \; = \; (H + C)(x) - (H + C)(x_e).$$

By assumption $\nabla^2(H + C)(x_e)$ is positive definite, so we know that x_e is a strict local minimum. Thus there exists a neighborhood U of x_e such that

$$V(x_e) \; = \; 0,$$

$$V(x) \; > \; 0, \quad \forall \, x \in U - \{x_e\}.$$

Since $H + C$ is a conserved quantity along trajectories of the given system, we have also

$$\dot{V}(x) \; = \; 0, \quad \forall \, x \in U - \{x_e\}.$$

We therefore conclude that x_e is Lyapunov stable by Lemma 5.2.1. A similar argument can be applied for the case that $\nabla^2(H + C)(x_e)$ is negative definite.

■

REMARK 5.2.3

   In Theorem 5.2.2, the Casimir function C could be any combination of smooth functions of the Casimir functions C_i in (5.2.3). Therefore we have a family of candidate

Lyapunov functions. Among them, we may vary some parameters to get a suitable Lyapunov function.

∎

This theorem provides us with a systematic method for determining stability of equilibria in noncanonical hamiltonian systems, cf. [33]. In particular, it helps in determining the relative stability modulo G corresponding to the projected Hamiltonian vector field $X_{\tilde{H}_L}$.

5.3. Lagrange-Multiplier Method

Now we describe an alternative approach to obtain a stability theorem. This is a scheme suggested by Maddocks in [44]. The stability of equilibrium of the Poisson system (5.2.2) is once again the subject of study. Here we *assume that the null space of* Λ *is one-dimensional.* Consider the constrained variational problem

$$\begin{aligned} \min \quad & H(x), \\ \text{subject to} \quad & C(x) = b, \end{aligned} \tag{5.3.1}$$

where b is a constant representing prescribed data and C is a particular Casimir function. The *Lagrangian* corresponding to this optimization problem can be written as

$$L(x, \lambda) = H(x) - \lambda C(x), \tag{5.3.2}$$

with $\lambda \in \mathbb{R}$. The first-order necessary conditions for (5.3.1) then coincide with (5.2.3) for $m = 1$. We now recall the following lemma.

LEMMA 5.3.1 (See e.g. Bertsekas [9], p. 68)

Let P be a symmetric matrix and Q a positive semidefinite symmetric matrix, both of dimension $n \times n$. Assume that, for $x \in \mathbb{R}^n$,

$$< x,\, Px > \; > \; 0, \quad \forall\, x \neq 0, \text{ such that } < x,\, Qx > \; = \; 0,$$

then there exists a (large, positive) scalar α such that

$$P + \alpha Q > 0,$$

namely, $P + \alpha Q$ is positive definite.

∎

We can now state the stability criterion as follows.

THEOREM 5.3.2

Suppose that x_e and $\lambda_e \in \mathbb{R}$ are such that

$$\nabla_x L(x_e, \lambda_e) = 0, \qquad\qquad (5.3.3a)$$

and, moreover,

$$< h, \; \nabla_x^2 L(x_e, \lambda_e) \, h > \; > \; 0, \quad \forall \, h \neq 0 \text{ such that } < \nabla C(x_e), \; h > \; = \; 0. \qquad (5.3.3b)$$

Then x_e is a Lyapunov stable equilibrium of (5.2.2).

Proof

Let

$$P \; = \; \nabla_x^2 L(x_e, \lambda_e),$$

$$Q \; = \; \nabla C(x_e) \, \nabla C(x_e)^T,$$

so that by hypothesis P and Q satisfy the conditions of the previous lemma. Thus we can find $\alpha \in \mathbb{R}$ such that $P + \alpha Q$ is a positive definite matrix. Now, with the notation $b = C(x_e)$, we define the *augmented Lagrangian* by,

$$
\begin{aligned}
L_\alpha(x, \lambda) \; &= \; H(x) \; - \; \lambda C(x) \; + \; \frac{1}{2}\alpha \left(C(x) - b \right)^2, \\
&= \; H(x) \; + \; \bar{C}(x).
\end{aligned}
$$

Then we have,

$$\nabla_x L_\alpha(x_e, \lambda_e) = \nabla H(x_e) + \lambda_e \nabla C(x_e) + \alpha \left(C(x_e) - b \right) \nabla C(x_e) \; = \; 0,$$

$$\nabla_x^2 L_\alpha(x_e, \lambda_e) = \nabla^2 H(x_e) + \left(\lambda_e + \alpha \big(C(x_e) - b \big) \right) \nabla^2 C(x_e) + \alpha \; \nabla C(x_e) \nabla C(x_e)^T$$

$$= P \; + \; \alpha \, Q \; > 0.$$

Since the function $\bar{C}$ is also a Casimir function, Theorem 5.2.2 can be applied to conclude that x_e is a Lyapunov stable equilibrium of system (5.2.2).

∎

93

REMARK 5.3.3

(a) Conditions (5.3.3) form a set of sufficient conditions for x_e to be a constrained local minimizer of (5.3.1).

(b) In an application of Theorem 5.2.2, we would search for a suitable Casimir C to fulfill the condition (5.2.4), cf. Remark 5.2.3. However, in the application of Theorem 5.3.2 we can fix a particular Casimir C and a scalar λ satisfying (5.3.3a), and then attempt to verify (5.3.3b). The analysis in Chapter 6 will illustrate the differences between the two schemes.

(c) With appropriate hypotheses, Theorem 5.3.2 can be generalized to cases in which there are n independent Casimirs, and the underlying space is infinite dimensional.

■

To apply the techniques of the current and previous sections to verify the relative stability modulo G of a relative equilibrium, we need to first find Casimir functions associated with the induced Poisson bracket $\{\cdot,\cdot\}_i$ defined in (3.3.5). For a hamiltonian G-space, there is a natural way to construct a family of Casimir functions from the momentum mapping J as follows (see, e.g. [62]).

Suppose $\phi : \mathcal{G}^* \to \mathbb{R}$ is an Ad^*-invariant function on the dual of the Lie algebra, namely, for $\mu \in \mathcal{G}^*$,

$$\phi(Ad_g^* \, \mu) \;=\; \phi(\mu), \qquad \forall \, g \in G. \tag{5.3.4}$$

We define a function

$$C_\phi \;=\; \phi \circ J \; : \; TQ \to \mathbb{R}. \tag{5.3.5}$$

It is easy to see that this is a G-invariant function. In fact,

$$C_\phi(\Phi_g^T(v_x)) \;=\; \phi \circ J \circ \Phi_g^T(v_x) \;=\; \phi \circ Ad_g^* \circ J(v_x) \;=\; \phi \circ J(v_x).$$

Thus C_ϕ induces a function $\tilde{C}_\phi$ on TQ/G through

$$\tilde{C}_\phi \circ \bar{\tau} \;=\; C_\phi. \tag{5.3.6}$$

Now we prove that $\tilde{C}_\phi$ is a Casimir function corresponding to the induced Poisson bracket $\{\cdot,\cdot\}_i$. For arbitrary $\tilde{f} \in \mathcal{F}(TQ/G)$, we have

$$\{\ \tilde{f},\ \tilde{C}_\phi\ \}_i \circ \tilde{\tau}\ =\ \{\ \tilde{f} \circ \tilde{\tau},\ \tilde{C}_\phi \circ \tilde{\tau}\ \}_L\ =\ \{\ f,\ C_\phi\ \}_L,$$

where f is a G-invariant function. Recall that J is an integral of any vector field induced from a G-invariant function, cf. Theorem 3.2.6. Thus

$$\{\ f,\ C_\phi\ \}_L\ =\ d(\phi \circ J) \cdot X_f\ =\ 0.$$

It follows that $\tilde{C}_\phi$ is a Casimir function with respect to the induced Poisson bracket. The dual-paring picture in Figure 5.3.1 illustrates the structures.
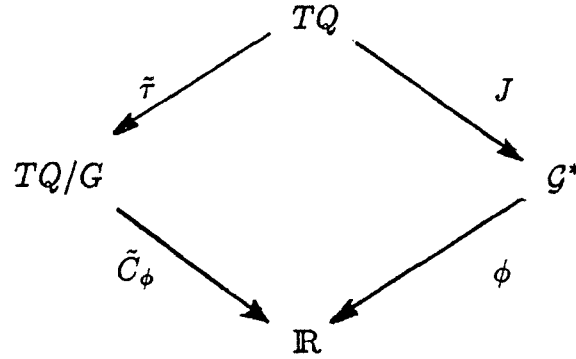
Figure 5.3.1. Dual-Pairing

For a gyroscopic system with symmetry, the momentum mapping can be found in (3.2.11). We could then construct Casimir functions through the above process. In particular, for the special case that $G = SO(3)$, the Ad^* action is, for $B \in SO(3)$, $\hat{\mu} \in so(3)^*$,

$$Ad^*_B\hat{\mu}\ =\ B\hat{\mu}B^T\ =\ \widehat{B\mu}.$$

The function defined by

$$\phi(\hat{\mu})\ =\ |\ \mu\ |, \tag{5.3.7}$$

95

where $|\cdot|$ denotes the euclidean norm, is an Ad^*-invariant function, since

$$\phi(Ad^*_B\hat{\mu}) = \phi(\widehat{B\mu}) = |B\mu| = |\mu|.$$

Consequently, from (5.3.5), the norm of the momentum mapping gives rise to a Casimir function for this special case. This applies to most mechanical systems with the symmetry of rotation group.

5.4. Energy Momentum Method

The previous two sections are concerned with relative stability modulo G of a relative equilibrium in TQ corresponding to the Poisson-reduced dynamics. In this section, the relative stability modulo G_μ will be examined based on the momentum mapping. Here we extend the energy-momentum method to the general framework of gyroscopic systems with symmetry. At generic points (in TQ/G), the two notions of relative stability are equivalent, cf. Theorem 5.1.3.

Let (P, ω) be a symplectic manifold on which the Lie group G acts symplectically and let $J : P \to \mathcal{G}^*$ be an Ad^*-equivariant momentum mapping for this action (see Section 3.1 for definitions). Assume we could perform symplectic reduction on P in the sense of Marsden and Weinstein [48]. The reduced phase space is denoted by $P_\mu = J^{-1}(\mu)/G_\mu$. Let $H : P \to \mathbb{R}$ be invariant under the action of G. It induces a Hamiltonian function H^μ on P_μ satisfying

$$H^\mu \circ \pi_\mu = H \circ i_\mu,$$

where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \hookrightarrow P$ is the inclusion map. We are interested in the stability property of a relative equilibrium under the reduced dynamics X_{H^μ} on the reduced space P_μ, or the relative stability modulo G_μ in $J^{-1}(\mu)$. By construction, H^μ is a first integral of the reduced dynamics. Thus if H^μ has a strict local minimum at $\pi_\mu(z_e)$ where z_e is a relative equilibrium, then H^μ serves as a Lyapunov function. Lemma 5.2.1 can be invoked to conclude stability. Since, for $z \in J^{-1}(\mu) \subset P$,

$$H^\mu(\pi_\mu(z)) \;=\; H(i_\mu(z)) \;=\; H\big|_{J^{-1}(\mu)}(z),$$

the condition for $\pi_\mu(z_e)$ being a strict local minimum of H^μ is equivalent to the condition for z_e being a strict local minimum of $H\big|_{J^{-1}(\mu)}$ modulo the tangent directions of the group orbit, $G_\mu \cdot z$. This in turn corresponds to checking that the relative equilibrium z_e solves the constrained minimization problem,

$$\min \quad H(z)$$

$$\text{subject to} \quad J(z) \;=\; \mu_e \;=\; J(z_e).$$

This problem could be further formulated as checking z_e to be a strict local minimum of $H - \langle J, \xi \rangle$ in all directions on $J^{-1}(\mu_e)$ except along the tangent directions to the group orbit generated by G_μ. These heuristic discussions could be formally spelled out in the following, which gives rise to the *energy-momentum method*, cf. [63], [62], [56], [61].

Define

$$H_\xi(z) \;=\; H(z) - \langle J(z),\, \xi \rangle. \tag{5.4.1}$$

From the relative equilibrium theorem, cf. Theorem 3.4.3, each relative equilibrium of the system is a critical point of H_ξ, for some $\xi \in \mathcal{G}$, namely,

$$DH_\xi(z_e) \cdot \delta z \;=\; 0, \quad \forall\, \delta z \;\in\; T_{z_e} P. \tag{5.4.2}$$

From previous discussions, the definiteness of the second variation of H_ξ on a subspace S of $T_{z_e} P$ satisfying

$$S \;\cong\; T_{z_e} J^{-1}(\mu_e) \,/\, T_{z_e}(G_\mu \cdot z_e), \tag{5.4.3}$$

implies the stability of the relative equilibrium z_e in the reduced dynamics. One way to find such a space S is to find a complement of $T_{z_e}(G_\mu \cdot z_e)$ in $T_{z_e} J^{-1}(\mu_e)$ such that

$$T_{z_e} J^{-1}(\mu_e) \;=\; S \;\oplus\; T_{z_e}(G_\mu \cdot z_e).$$

Since $T_{z_e} J^{-1}(\mu_e) \;=\; \mathrm{Ker}\; DJ(z_e)$, which is the kernel of the operator $DJ(z_e)$, we could summarize the energy momentum method for relative stability as follows.

ALGORITHM 5.4.1   (Energy-Momentum Method)

    0. Pick $\xi \in \mathcal{G}$.

1. Solve the problem

$$DH_\xi(z) \cdot \delta z = 0, \quad \forall \, \delta z \in T_z P,$$

for a relative equilibrium z_e.

2. Compute $\mu_e = J(z_e)$ and determine the space Ker $DJ(z_e)$.

3. Find $S \subset$ Ker $DJ(z_e)$ such that

$$\text{Ker } DJ(z_e) = S \oplus T_{z_e}(G_\mu \cdot z_e).$$

4. Check the second variation of H_ξ on S. Definiteness of the second variation implies stability.

■

For visualizing the geometric pictures, see Figure 5.4.1.
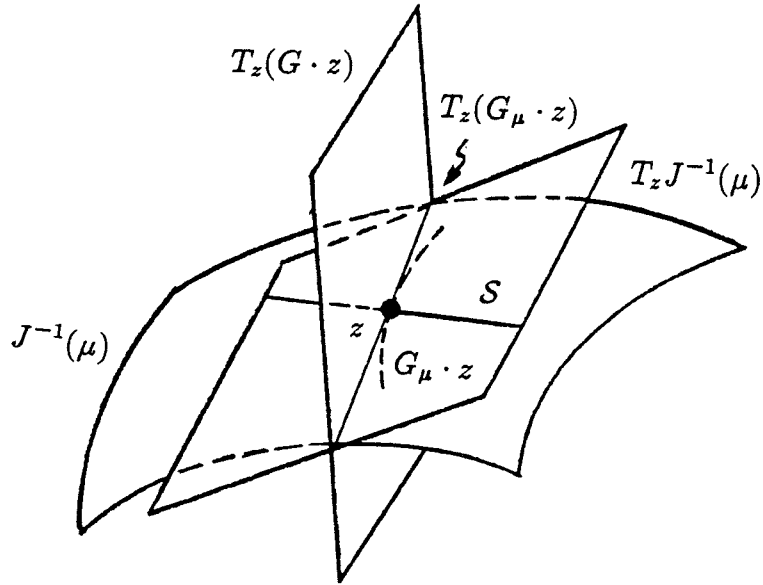
Figure 5.4.1. Energy-Momentum Method

Now we consider gyroscopic systems with symmetry introduced in Chapter 3. The underlying space is $P = TQ$ with the symplectic structure Ω_L. In this setting, the momentum mapping is given by, cf. (3.2.11),

98

$$J(v_x)(\xi) \;=\; \ll v_x + Y(x),\; \xi_Q(x) \gg_x, \tag{5.4.4}$$

and the energy-momentum functional is, cf. (3.4.3),

$$H_\xi(v_x) \;=\; K_\xi(v_x) \;+\; V_\xi(x), \tag{5.4.5}$$

where, cf. (3.4.5),

$$K_\xi(v_x) \;=\; \frac{1}{2} \ll v_x - \xi_Q(x),\; v_x - \xi_Q(x) \gg_x,$$
$$V_\xi(x) \;=\; V(x) - \ll Y(x),\; \xi_Q(x) \gg_x -\frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg_x . \tag{5.4.6}$$

We define the space

$$\mathcal{N}_x \;\triangleq\; \{\; \eta_Q(x) \;:\; \eta \in \mathcal{G} \;\}, \tag{5.4.7}$$

which is a subspace of $T_x Q$, and thus

$$\mathcal{N} \;\triangleq\; \cup_{x \in Q} \; \mathcal{N}_x, \tag{5.4.8}$$

is a subbundle of TQ. We could then decompose $T_x Q$ into $\mathcal{N}_x$ and $\mathcal{N}_x^\perp$ where $\mathcal{N}_x^\perp$ is the orthogonal complement of $\mathcal{N}_x$ with respect to the inner product associated with the riemannian metric. Every element $v \in T_x Q$ can be thus written uniquely as

$$v \;=\; \eta_Q(x) \;+\; \bar{v},$$

for $\eta \in \mathcal{G}$, $\bar{v} \in \mathcal{N}_x^\perp$. With this decomposition, the function K_ξ could be further written as, from (5.4.6),

$$K_\xi(x,\; v) \;=\; \frac{1}{2} \| \eta_Q(x) \;-\; \xi_Q(x) \|^2 \;+\; \frac{1}{2} \| \bar{v} \|^2. \tag{5.4.9}$$

Note that at relative equilibrium $(x_e,\; \xi_Q(x_e))$, we have $v_e = \xi_Q(x_e)$, and $\bar{v} = 0$. Thus the second term in K_ξ is nonnegative and vanishes at relative equilibrium with a positive semi-definite second variation. Define

$$\tilde{H}_\xi(x,\eta) \;=\; \frac{1}{2} \| \eta_Q(x) \;-\; \xi_Q(x) \|^2 \;+\; V_\xi(x). \tag{5.4.10}$$

We can determine the relative stability from this modified function on the space of $Q \times \mathcal{G}$. Taking variations in the space $Q \times \mathcal{G}$ corresponds to taking variations in the subbundle $\mathcal{N} \subset TQ$. We define the embedding

$$\mathcal{Z} : Q \times \mathcal{G} \to TQ,$$

$$(x, \eta) \mapsto (x, \eta_Q(x)).$$

Then $\bar{H}_\xi = H_\xi \circ \mathcal{Z}$. Similarly, define the *pre-momentum mapping*

$$\bar{J}(x, \eta) = J \circ \mathcal{Z}(x, \eta) = J(x, \eta_Q(x)). \qquad (5.4.11)$$

We may define an action on the space $Q \times \mathcal{G}$ as

$$\Psi : G \times (Q \times \mathcal{G}) \to Q \times \mathcal{G}$$

$$(g, (x, \eta)) \mapsto (g \cdot x, Ad_g \eta), \qquad (5.4.12)$$

where Ad is the adjoint action defined in (3.1.10). It can be shown that

$$\Phi_g^T \circ \mathcal{Z} = \mathcal{Z} \circ \Psi_g. \qquad (5.4.13)$$

Also we have

<u>LEMMA 5.4.2</u>

The pre-momentum mapping $\bar{J} : Q \times \mathcal{G} \to \mathcal{G}^*$ is Ad^*-equivariant.

Proof

    We have

$$\bar{J} \circ \Psi_g = J \circ \mathcal{Z} \circ \Psi_g = J \circ \Phi_g^T \circ \mathcal{Z}$$

$$= Ad_{g^{-1}}^* \circ J \circ \mathcal{Z} = Ad_{g^{-1}}^* \circ \bar{J}.$$

    ■

From this Lemma, it can be shown that the level set $\bar{J}^{-1}(\mu)$ is invariant under the action of the isotropy subgroup G_μ. Furthermore, let $\bar{H} : Q \times \mathcal{G} \to \mathbb{R}$ be defined as $\bar{H} = H \circ \mathcal{Z}$. From (5.4.13), the function $\bar{H}$ is invariant under the group action Ψ. The functional $\bar{H}_\xi$ can be now written as

$$\bar{H}_\xi = \bar{H} - \langle \bar{J}, \xi \rangle.$$

By the invariance properties, the restriction of $\bar{H}_\xi$ on $\bar{J}^{-1}(\mu)$,

$$\bar{H}_\xi \big|_{\bar{J}^{-1}(\mu)} = \bar{H} \big|_{\bar{J}^{-1}(\mu)} - \langle \mu, \xi \rangle.$$

is invariant under the group action of G_μ. As a consequence, the geometric picture is similar to Figure 5.4.1. An algorithm similar to Algorithm 5.4.1 can then be applied to check if (x_e, ξ) is a local minimizer of $\tilde{H}_\xi$ restricted to $\tilde{J}^{-1}(\mu)$. Before doing that, we introduce a few notations. The riemannian metric restricted to the subspace $\mathcal{N}_x$ provides an x-dependent bilinear form on the Lie algebra $\mathcal{G}$. This, in turn, induces a locked inertia tensor associated to $x \in Q$,

$$\mathbf{I}_{lock}(x) \; : \; \mathcal{G} \; \rightarrow \; \mathcal{G}^*,$$

defined through

$$\langle \xi, \; \mathbf{I}_{lock}(x)\eta \rangle \; \stackrel{\triangle}{=} \; \ll \xi_Q(x), \; \eta_Q(x) \gg_x, \tag{5.4.14}$$

for $\xi, \; \eta \in \mathcal{G}$. From the symmetry property of the riemannian metric, we have

$$\langle \xi, \; \mathbf{I}_{lock}(x)\eta \rangle \; = \; \langle \mathbf{I}_{lock}(x)\xi, \; \eta \rangle,$$

namely, $\mathbf{I}_{lock}(x)$ is symmetric. Also, we assume that, at x, the locked inertia tensor has an inverse,

$$\mathbf{I}_{lock}(x)^{-1} \; : \; \mathcal{G}^* \; \rightarrow \; \mathcal{G}.$$

On the other hand, the gyroscopic field also induces for each $x \in Q$ an element $\mathbf{I}_Y(x)$ in $\mathcal{G}^*$ defined by,

$$\langle \mathbf{I}_Y(x), \; \eta \rangle \; \stackrel{\triangle}{=} \; \ll Y(x), \; \eta_Q(x) \gg_x, \qquad \forall \; \eta \in \mathcal{G}. \tag{5.4.15}$$

We refer to $\mathbf{I}_Y(x)$ as the (x-dependent) *gyro-momentum*. The function $\tilde{H}_\xi$ may now be expressed as

$$\begin{aligned}
\tilde{H}_\xi(x, \; \eta) &= \frac{1}{2} \langle \eta - \xi, \; \mathbf{I}_{lock}(x)(\eta - \xi) \rangle \\
&\quad + V(x) \; - \; \frac{1}{2} \langle \xi, \; \mathbf{I}_{lock}(x)\xi \rangle \; - \; \langle \mathbf{I}_Y(x), \; \xi \rangle, \tag{5.4.16} \\
&= \frac{1}{2} \langle \eta - \xi, \; \mathbf{I}_{lock}(x)(\eta - \xi) \rangle \; + \; V_\xi(x)
\end{aligned}$$

with the pre-momentum mapping, from (5.4.4), (5.4.11), for $\eta \in \mathcal{G}$,

$$\langle \tilde{J}(x, \eta), \zeta \rangle = \langle J(x, \eta_Q(x)), \zeta \rangle$$

$$= \ll \eta_Q(x), \zeta_Q(x) \gg_x + \ll Y(x), \zeta_Q(x) \gg_x$$

$$= \langle \mathrm{I}_{lock}(x)\eta, \zeta \rangle + \langle \mathrm{I}_Y(x), \zeta \rangle,$$

or we may write

$$\tilde{J}(x, \eta) = \mathrm{I}_{lock}(x)\eta + \mathrm{I}_Y(x). \tag{5.4.17}$$

For $\mu \in \mathcal{G}^*$, the associated isotropy subalgebra $\mathcal{G}_{\mu_e}$ is defined in (3.1.13). With the inner product induced on $\mathcal{G}$ by the locked inertia tensor at x_e, we define the orthogonal complement of $\mathcal{G}_{\mu_e}$ to be

$$\mathcal{G}_{\mu_e}^\perp \triangleq \{\ \zeta \in \mathcal{G}\ :\ \langle \zeta, \mathrm{I}_{lock}(x_e)\eta \rangle = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\ \}. \tag{5.4.18}$$

Following the notations used in [61], we define the maps

$$\tilde{A} : \mathcal{G} \to \mathcal{G}^*, \quad \mathcal{A} : \mathcal{G} \to \mathcal{G},$$

by

$$\tilde{A}(\eta) \triangleq ad_\eta^* \mu_e, \quad \mathcal{A}(\eta) \triangleq \mathrm{I}_{lock}(x_e)^{-1} \tilde{A}(\eta), \tag{5.4.19}$$

respectively. As proved in [61], we have the following lemma.

<u>LEMMA 5.4.3</u>

Provided that $\mathcal{G}_{\mu_e}^\perp$ is finite dimensional or $\mathcal{A}$ is elliptic with respect to the inner product induced by $\mathrm{I}_{lock}(x_e)$, we have

   (i) $\mathcal{A}$ maps $\mathcal{G}$ onto $\mathcal{G}_{\mu_e}^\perp$.

   (ii) $\tilde{A}$ maps $\mathcal{G}$ onto $\mathcal{G}_{\mu_e}^a \subset \mathcal{G}^*$, where

$$\mathcal{G}_{\mu_e}^a = \{\ \mu \in \mathcal{G}^*\ :\ \langle \mu, \eta \rangle = 0,\ \forall\ \eta \in \mathcal{G}_{\mu_e}\ \},$$

is the annihilator of $\mathcal{G}_{\mu_e}$.

■

With these notations, we are now ready to apply Algorithm 5.4.1. We proceed as follows.

Step 0. Fix $\xi \in \mathcal{G}$.

Step 1. It is straightforward to derive

$$D\tilde{H}_\xi(x,\ \eta)\ (\delta x,\ \delta\eta)$$
$$=\ DV_\xi(x)\delta x\ +\ \langle\delta\eta,\ \mathbf{I}_{lock}(x)(\eta-\xi)\rangle\ +\ \frac{1}{2}\langle\eta-\xi,\ (D\mathbf{I}_{lock}(x)\delta x)(\eta-\xi)\rangle.$$

The relative equilibrium is given by the conditions

$$DV_\xi(x_e)\ =\ 0,\quad \eta_e\ =\ \xi,$$

which match with the conditions we obtained in Algorithm 3.4.4.

Step 2.

For the relative equilibrium $(x_e,\ \xi)$, we have

$$\mu_e\ =\ \tilde{J}(x_e,\ \xi)\ =\ \mathbf{I}_{lock}(x_e)\xi\ +\ \mathbf{I}_Y(x_e). \tag{5.4.20}$$

Now we find the space $\operatorname{Ker} D\tilde{J}(x_e,\ \xi)$. From (5.4.17),

$$D\tilde{J}(x,\ \eta)(\delta x,\ \delta\eta)\ =\ \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\ \tilde{J}(x+\epsilon\,\delta x,\ \eta+\epsilon\,\delta\eta)$$
$$=\ \big(D\mathbf{I}_{lock}(x)\delta x\big)\eta\ +\ \mathbf{I}_{lock}(x)\delta\eta\ +\ D\mathbf{I}_Y(x)\delta x. \tag{5.4.21}$$

Here $x + \epsilon\,\delta x$ denotes the integral curve corresponding to the tangent vector δx at x. For $(\delta x,\ \delta\eta)$ to be in $\operatorname{Ker} D\tilde{J}(x_e,\ \xi)$, we must have, from (5.4.21),

$$\delta\eta\ =\ -\mathbf{I}_{lock}(x_e)^{-1}\big((D\mathbf{I}_{lock}(x_e)\delta x)\xi\ +\ D\mathbf{I}_Y(x_e)\delta x\big),$$
$$=\ \mathbf{I}_{lock}(x_e)^{-1}\ \mathtt{ident}^Y_\xi(x_e)\delta x. \tag{5.4.22}$$

where the map $\mathtt{ident}^Y\colon \mathcal{G} \times TQ \to \mathcal{G}^*$ is defined by, for $(x, \delta x) \in TQ$,

$$\mathtt{ident}^Y_\xi(x)\delta x\ \triangleq\ -\left((D\mathbf{I}_{lock}(x)\delta x)\xi\ +\ D\mathbf{I}_Y(x)\delta x\right). \tag{5.4.23}$$

This map specializes to the map $ident_\xi$ defined in [61] when $Y = 0$, i.e., for simple mechanical systems with symmetry. The properties of this map play an important role in our subsequent development. We need the following lemma.

LEMMA 5.4.4

For $x \in Q$ and $\zeta,\ \nu,\ \eta \in \mathcal{G}$, we have the following identities,

$$\langle \zeta, \ (D\mathbf{I}_{lock}(x)\eta_Q(x))\nu \rangle \ = \ \langle [\zeta, \ \eta], \ \mathbf{I}_{lock}(x)\nu \rangle \ + \ \langle [\nu, \ \eta], \ \mathbf{I}_{lock}(x)\zeta \rangle, \qquad (5.4.24)$$

$$\langle \zeta, \ D\mathbf{I}_Y(x)\eta_Q(x) \rangle \ = \ \langle \mathbf{I}_Y(x), \ [\zeta, \ \eta] \rangle. \qquad (5.4.25)$$

Proof

The proof of (5.4.24) could be found in [61]. Here we only verify (5.4.25). By definition (5.4.15),

$$
\begin{aligned}
\langle \zeta, \ D\mathbf{I}_Y(x)\eta_Q(x) \rangle \ &= \ \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \langle \mathbf{I}_Y(\exp \epsilon \, \eta \cdot x), \ \zeta \rangle \\
&= \ \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \ll Y(\exp \epsilon \, \eta \cdot x), \ \zeta_Q(\exp \epsilon \, \eta \cdot x) \gg_{\exp \epsilon \, \eta \cdot x}, \\
&= \ L_{\eta_Q} \ll Y, \ \zeta_Q \gg \ (x), \\
&= \ \ll L_{\eta_Q} Y(x), \ \zeta_Q(x) \gg_x \ + \ \ll Y(x), \ L_{\eta_Q}\zeta_Q(x) \gg_x,
\end{aligned}
$$

by using Lemma 3.1.2. Also we have the identity $L_{\eta_Q}\zeta_Q \ = \ [\zeta, \ \eta]_Q$. With (3.1.19) in Lemma 3.1.1, it follows that

$$
\begin{aligned}
\langle \zeta, \ D\mathbf{I}_Y(x)\eta_Q(x) \rangle \ &= \ \ll Y(x), \ [\zeta, \ \eta]_Q(x) \gg_x, \\
&= \ \langle \mathbf{I}_Y(x), \ [\zeta, \ \eta] \rangle.
\end{aligned}
$$

This is the desired identity.

∎

We now evaluate the map $\mathbf{ident}^Y_\xi(x_e)$ restricted to the space $\mathcal{N}_{x_e}$.

LEMMA 5.4.5

For $\eta \in \mathcal{G}$, at relative equilibrium $(x_e, \ \xi)$,

$$\mathbf{ident}^Y_\xi(x_e)\,\eta_Q(x_e) \ = \ ad^*_\eta \, \mu_e \ + \ \mathbf{I}_{lock}(x_e)[\eta, \ \xi]. \qquad (5.4.26)$$

Proof

From the definition (5.4.23), for arbitrary $\nu \in \mathcal{G}$,

$$\langle \mathbf{ident}^Y_\xi(x_e)\eta_Q(x_e), \ \nu \rangle \ = \ -\langle \nu, \ (D\mathbf{I}_{lock}(x_e)\eta_Q(x_e))\xi \rangle \ - \ \langle D\mathbf{I}_Y(x_e)\eta_Q(x_e), \ \nu \rangle,$$

From Lemma 5.4.4, this could be further written as

$$-\langle[\nu,\ \eta],\ \mathbf{I}_{lock}(x_e)\xi\rangle\ -\ \langle[\xi,\ \eta],\ \mathbf{I}_{lock}(x_e)\nu\rangle - \langle\mathbf{I}_Y(x_e),\ [\nu,\ \eta]\rangle$$

$$=\ \langle\mathbf{I}_{lock}(x_e)\xi+\mathbf{I}_Y(x_e),\ [\eta,\ \nu]\rangle\ +\ \langle\mathbf{I}_{lock}(x_e)[\xi,\ \eta],\ \nu\rangle,$$

$$=\ \langle ad_\eta^*\mu_e,\ \nu\rangle\ +\ \langle\mathbf{I}_{lock}(x_e)[\xi,\ \eta],\ \nu\rangle.$$

where the formula for μ_e in (5.4.20) has been used. We thus established (5.4.26).

∎

The discussions in Step 2. could be summarized by writing

$$\mathrm{Ker}\,D\tilde{J}(x_e,\xi)\ =\ \{\ (\delta x,\ \eta)\in T_{(x_e,\xi)}(Q\times\mathcal{G})\ :$$
$$\eta\ =\ \mathbf{I}_{lock}(x_e)^{-1}\ \mathbf{ident}^Y_\xi(x_e)\,\delta x\ \}. \tag{5.4.27}$$

Step 3.

As shown in (5.4.27), the component of $\mathcal{G}$ in $\mathrm{Ker}\,D\tilde{J}(x_e,\xi)$ is determined from the variation δx in $T_{x_e}Q$. We thus only need to decompose the kernel space with respect to $T_{x_e}(G_{\mu_e}\cdot x_e)$. Since

$$\mathcal{N}^{\mu_e}_x\ \triangleq\ T_{x_e}(G_{\mu_e}\cdot x_e)\ =\ \{\ \eta_Q(x_e)\in T_{x_e}Q\ :\ \eta\in\mathcal{G}_{\mu_e}\ \}, \tag{5.4.28}$$

we may find the orthogonal complement of it with respect to the riemannian metric as,

$$\mathcal{V}\ =\ \{\ \delta x\in T_{x_e}Q\ :\ \ll\delta x,\ \eta_Q(x_e)\gg_{x_e}\ =\ 0,\ \ \forall\,\eta\in\mathcal{G}_{\mu_e}\ \} \tag{5.4.29}$$

Consequently, we may write the space $\tilde{\mathcal{S}}$ as

$$\tilde{\mathcal{S}}\ =\ \{\ (\delta x,\ \eta)\in\mathcal{V}\times\mathcal{G}\ :\ \eta\ =\ \mathbf{I}_{lock}(x_e)^{-1}\ \mathbf{ident}^Y_\xi(x_e)\,\delta x\ \}. \tag{5.4.30}$$

and we get $\mathrm{Ker}\,D\tilde{J}(x_e,\xi)\ =\ \tilde{\mathcal{S}}\ \oplus\ T_{(x_e,\xi)}(G_{\mu_e}\cdot(x_e,\xi))$, where, with respect to the action Ψ defined in (5.4.12),

$$T_{(x_e,\xi)}(G_{\mu_e}\cdot(x_e,\xi))\ =\ \{\ (\zeta_Q(x_e),\ ad_\zeta\xi)\ :\ \zeta\in\mathcal{G}_{\mu_e}\ \},$$

can be shown to be a subspace of $\mathrm{Ker}\,D\tilde{J}(x_e,\xi)$.

Step 4.

Now we check the definiteness of the second variation of $\bar{H}_\xi$ on the space $\tilde{\mathcal{S}}$. The block diagonalization techniques prove to be useful in this context. We introduce a decomposition of the space $\mathcal{V}$ as follows. Let

$$\mathcal{V}_{RIG} = \{ \zeta_Q(x_e) \; : \; \zeta \in \mathcal{G}_{\mu_e}^{\perp} \}, \tag{5.4.31}$$

$$\mathcal{V}_{INT} = \{ \delta x \in \mathcal{V} \; : \; \langle \zeta, \; \mathtt{ident}^Y_{\xi}(x_e)\delta x \rangle = 0, \quad \forall \; \zeta \in \mathcal{G}_{\mu_e}^{\perp} \}. \tag{5.4.32}$$

By definitions (5.4.7), (5.4.28) and (5.4.31), we have

$$\mathcal{N}_x = \mathcal{N}_x^{\mu_e} \oplus \mathcal{V}_{RIG}.$$

But the condition for

$$\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}, \tag{5.4.33}$$

will be discussed further in the following (cf. Lemma 5.4.7). The relationship between these spaces is depicted in Figure 5.4.2.
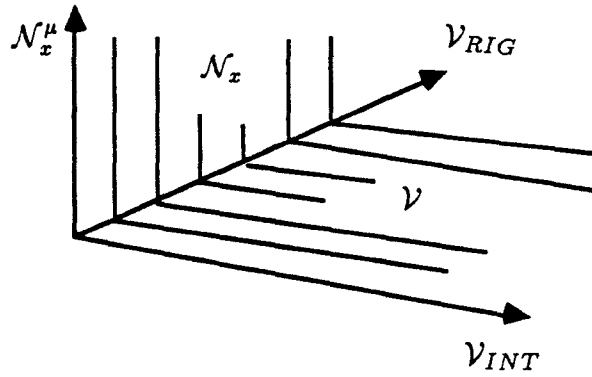
Figure 5.4.2. Decomposition of $T_x Q$

The second variation of $\bar{H}_{\xi}$ could be found as

$$D^2 \bar{H}_{\xi}(x_e, \xi) \cdot (\delta x_1, \; \eta_1) \cdot (\delta x_2, \; \eta_2)$$

$$= \langle \eta_1, \; \mathbf{I}_{lock}(x_e)\eta_2 \rangle + D^2 V_{\xi}(x_e) \cdot \delta x_1 \cdot \delta x_2,$$

$$= \langle \mathtt{ident}^Y_{\xi}(x_e)\delta x_1, \; \mathbf{I}_{lock}(x_e)^{-1} \, \mathtt{ident}^Y_{\xi}(x_e)\delta x_2 \rangle + D^2 V_{\xi}(x_e) \cdot \delta x_1 \cdot \delta x_2,$$

for $(\delta x_1, \; \eta_1), \; (\delta x_2, \; \eta_2) \in \mathrm{Ker}\, D\tilde{J}(x_e, \xi)$. A bilinear form on $T_{x_e}Q \times T_{x_e}Q$ is defined as,

$$\mathbf{B}_{\xi}(\delta x_1, \; \delta x_2) \; \overset{\triangle}{=} \; \langle \mathtt{ident}^Y_{\xi}(x_e)\delta x_1, \; \mathbf{I}_{lock}(x_e)^{-1} \mathtt{ident}^Y_{\xi}(x_e)\delta x_2 \rangle$$
$$+ \; D^2 V_{\xi}(x_e) \cdot \delta x_1 \cdot \delta x_2. \tag{5.4.34}$$

Note that

$$D^2 \tilde{H}_\xi(x_e, \xi) \cdot (\delta x_1, \eta_1) \cdot (\delta x_2, \eta_2) = \mathbf{B}_\xi(\delta x_1, \delta x_2).$$

We have the following key proposition.

PROPOSITION 5.4.6

For $\eta_Q(x_e) \in \mathcal{V}_{RIG}$, and $\delta x \in \mathcal{V}_{INT}$,

$$\mathbf{B}_\xi(\eta_Q(x_e), \delta x) = 0.$$

Proof

We first find the second variation of V_ξ. By the property that V is G-invariant and Lemma 5.4.4, we have

$$
\begin{aligned}
& DV_\xi(x) \cdot \eta_Q(x) \\
&= DV(x) \cdot \eta_Q(x) - \frac{1}{2} \langle \xi, \left(D\mathbf{I}_{lock}(x) \cdot \eta_Q(x)\right)\xi \rangle - \langle D\mathbf{I}_Y(x) \cdot \eta_Q(x), \xi \rangle, \\
&= - \langle [\xi, \eta], \mathbf{I}_{lock}(x)\xi + \mathbf{I}_Y(x) \rangle. \qquad (5.4.35)
\end{aligned}
$$

It is then easy to see that, cf. (5.4.23),

$$
\begin{aligned}
D^2 V_\xi(x) \cdot \eta_Q(x) \cdot \delta x &= -\langle [\xi, \eta], (D\mathbf{I}_{lock}(x) \cdot \delta x)\xi + D\mathbf{I}_Y(x) \cdot \delta x \rangle. \\
&= \langle [\xi, \eta], \text{ident}^Y_\xi(x)\delta x \rangle. \qquad (5.4.36)
\end{aligned}
$$

Next we evaluate the bilinear form on $\mathcal{V}_{RIG} \times \mathcal{V}_{INT}$. Combining (5.4.34), (5.4.36) and using Lemma 5.4.5, we get,

$$
\begin{aligned}
& \mathbf{B}_\xi(\eta_Q(x_e), \delta x) \\
&= \langle \text{ident}^Y_\xi(x_e) \eta_Q(x_e), \mathbf{I}_{lock}(x_e)^{-1} \text{ident}^Y_\xi(x_e) \delta x \rangle + \langle [\xi, \eta], \text{ident}^Y_\xi(x_e) \delta x \rangle, \\
&= \langle ad^*_\eta \mu_e + \mathbf{I}_{lock}(x_e)[\eta, \xi], \mathbf{I}_{lock}(x_e)^{-1} \text{ident}^Y_\xi(x_e) \delta x \rangle + \langle [\xi, \eta], \text{ident}^Y_\xi(x_e) \delta x \rangle, \\
&= \langle ad^*_\eta \mu_e, \mathbf{I}_{lock}(x_e)^{-1} \text{ident}^Y_\xi(x_e) \delta x \rangle, \\
&= \langle \mathcal{A}(\eta), \text{ident}^Y_\xi(x_e) \delta x \rangle, \qquad (5.4.37)
\end{aligned}
$$

where $\mathcal{A}$ is defined in (5.4.19). From Lemma 5.4.3, $\mathcal{A}(\eta) \in \mathcal{G}_{\mu_e}^\perp$. For $\delta x \in \mathcal{V}_{INT}$, by the definition of $\mathcal{V}_{INT}$, cf. (5.4.32), the desired property follows.

∎

With this proposition, the second variation of $\tilde{H}_\xi$ on $\tilde{S}$ at relative equilibrium is diagonalized into two blocks. Checking the definiteness of $D^2 \tilde{H}_\xi$ on $\tilde{S}$ is thus equivalent to checking the definiteness of $\mathbf{B}_\xi$ on the spaces of $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ and $\mathcal{V}_{INT} \times \mathcal{V}_{INT}$ independently, under the assumption that (5.4.33) holds. These techniques simplify the computations quite significantly, as will be seen in later chapters. In particular, the form of $\mathbf{B}_\xi$ on $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ could be worked out explicitly. From (5.4.37),

$$
\begin{aligned}
&\mathbf{B}_\xi(\eta_Q(x_e),\ \eta_Q(x_e)) \\
&= \langle \mathcal{A}(\eta),\ \mathtt{ident}^Y{}_\xi(x_e)\,\eta_Q(x_e) \rangle, \\
&= \langle \mathcal{A}(\eta),\ ad_\eta^* \mu_e + \mathbf{I}_{lock}(x_e)[\eta,\ \xi] \rangle, \\
&= \langle ad_\eta^* \mu_e,\ \mathbf{I}_{lock}(x_e)^{-1}\, ad_\eta^* \mu_e \rangle\ +\ \langle ad_\eta^* \mu_e,\ ad_\eta \xi \rangle.
\end{aligned}
\tag{5.4.38}
$$

This is the *Arnold block* analogous to the one in simple mechanical systems with symmetry [62]. *The gyro-momentum is buried in μ_e and can affect definiteness of this block.* Definiteness of this block ensures the decomposition (5.4.33) of the space $\mathcal{V}$, which is proved in the following lemma.

LEMMA 5.4.7

Positive definiteness of $\mathbf{B}_\xi$ on $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ implies that $\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}$.

Proof

We only consider here the finite dimensional case. For the infinite dimensional case, the proof is analogous to the one in [61]. Letting $\zeta_Q(x_e) \in \mathcal{V}_{RIG} \cap \mathcal{V}_{INT}$, we have $\zeta \in \mathcal{G}_{\mu_e}^\perp$, and

$$
\langle \nu,\ \mathtt{ident}^Y{}_\xi(x_e)\zeta_Q(x_e) \rangle\ =\ 0, \quad \forall\, \nu \in \mathcal{G}_{\mu_e}^\perp.
\tag{5.4.39}
$$

We choose, in (5.4.39),

$$
\nu\ =\ \mathcal{A}(\zeta)\quad \in \mathcal{G}_{\mu_e}^\perp,
$$

which is ensured by Lemma 5.4.3. By comparing with (5.4.38), we get

$$B_\xi\big(\zeta_Q(x_e),\ \zeta_Q(x_e)\big)\ =\ 0.$$

Since, by assumption, B_ξ is positive definite, this implies $\zeta\ =\ 0$. Namely,

$$\mathcal{V}_{RIG}\ \cap\ \mathcal{V}_{INT}\ =\ \{\ 0\ \}.$$

On the other hand, $\dim \mathcal{V}_{RIG}\ +\ \dim \mathcal{V}_{INT}\ =\ \dim \mathcal{V}$. Thus the decomposition (5.4.33) holds.

∎

With this Lemma, we don't need to verify (5.4.33) explicitly. It is guaranteed by checking the definiteness of the Arnold block. We summarize the discussion in this step in the following theorem.

THEOREM 5.4.8

If the bilinear form B_ξ is positive definite on both $\mathcal{V}_{RIG}\times\mathcal{V}_{RIG}$ and $\mathcal{V}_{INT}\times\mathcal{V}_{INT}$, then the relative equilibrium $(x_e,\ \xi_Q(x_e))\in TQ$ is relatively stable modulo G_μ.

∎

Now we have completed the process of Algorithm 5.4.1 of determining the relative stability for the gyroscopic systems with symmetry. The block diagonalization of the second variation of $\tilde{H}_\xi$ is achieved on the constrained subspace $\tilde{S}$. A few explanatory remarks follow. First, we note a necessary condition for relative equilibrium.

PROPOSITION 5.4.9

At relative equilibrium $(x_e,\ \xi)$,

$$ad^*_\xi\,\mu_e\ =\ 0. \tag{5.4.40}$$

Proof

This result holds in the general setting of hamiltonian systems with symmetry, see Proposition 1.2 of [61] for a related proof. Here we give a proof applicable to the setting of gyroscopic systems with symmetry.

From the discussion in Step 1, at relative equilibrium, $DV_\xi = 0$. We evaluate it along the directions in $\mathcal{N}_{x_e}$ with the formula (5.4.35). For all $\eta\in\mathcal{G}$,

$$0 = DV_\xi(x_e) \cdot \eta_Q(x_e) = -\langle [\xi, \eta], \mathbf{I}_{lock}(x_e)\xi + \mathbf{I}_Y(x_e) \rangle,$$

$$= -\langle ad_\xi \eta, \mu_e \rangle = -\langle \eta, ad_\xi^* \mu_e \rangle,$$

from (5.4.20), (3.1.11). Thus, at relative equilibrium, we have $ad_\xi^* \mu_e = 0$.

∎

Next we consider the amended potential introduced for simple mechanical systems with symmetry. From (5.4.17), we may construct a mapping from $Q \times \mathcal{G}^*$ to $Q \times \mathcal{G}$ as

$$(x, \mu) \mapsto \left(x, \mathbf{I}_{lock}(x)^{-1} (\mu - \mathbf{I}_Y(x)) \right). \tag{5.4.41}$$

With this transformation, we may write the function $\tilde{H}_\xi$ on the space $Q \times \mathcal{G}^*$ as, from (5.4.16),

$$\tilde{\tilde{H}}_\xi(x, \mu) = V_\mu(x) - \langle \mu, \xi \rangle,$$

where

$$V_\mu(x) = V(x) + \frac{1}{2} \langle \mu - \mathbf{I}_Y(x), \mathbf{I}_{lock}(x)^{-1} (\mu - \mathbf{I}_Y(x)) \rangle, \tag{5.4.42}$$

is called the *amended potential*. It can be shown that, at relative equilibrium (x_e, ξ), we have

$$DV_\mu(x_e)\, \delta x = DV_\xi(x_e)\, \delta x,$$

$$D^2 V_\mu(x_e) \cdot \delta x_1 \cdot \delta x_2 = \mathbf{B}_\xi(\delta x_1, \delta x_2). \tag{5.4.43}$$

Thus the stability conditions in Theorem 5.4.8 are equivalent to the conditions for the relative equilibrium to be a constrained strict local minimizer of the function V_μ. This conclusion is analogous to the Lagrange-Dirichlet theorem in spirit [6]. We phrase it as a theorem.

THEOREM 5.4.10

For gyroscopic systems with symmetry, the components of relative equilibria in the configuration space are the critical points of the function V_μ. If the configuration component x_e of a relative equilibrium is a constrained strict local minimizer of the function V_μ (i.e. by taking out the neutral directions tangent to $G_{\mu_e} \cdot x_e$), then the relative equilibrium is relatively stable modulo G_{μ_e}.

∎

REMARK 5.4.11

In most practical problems, the augmented potential V_ξ is easier to compute than the amended potential V_μ. From Theorem 5.4.8 and (5.4.34), it is clear that positive-definiteness of the second variation of V_ξ on $\mathcal{V}$ is sufficient for stability. Following arguments similar to the discussion regarding V_μ, we could get an analogous statement as in Theorem 5.4.10 with V_μ replaced by the augmented potential V_ξ.

∎

In the following, we consider two special cases. First, it is easy to see that for the case of $Q = G$, cf. Figure 5.4.2,

$$\mathcal{V}_{INT} = \{\, 0 \,\}.$$

Consequently, we only need to consider the Arnold block for stability. Secondly, for the case of $G = SO(3)$, we have

$$ad_{\hat{\xi}}\, \hat{\eta} = \widehat{\xi \times \eta}, \tag{5.4.44a}$$

$$ad_{\hat{\xi}}^*\, \hat{\mu} = \widehat{\mu \times \xi}. \tag{5.4.44b}$$

where $\hat{\xi}$, $\hat{\eta} \in so(3)$, and $\hat{\mu} \in so^*(3)$. Thus, Proposition 5.4.9 implies

$$\mu_e \times \xi = 0, \quad \text{or} \quad \mu_e = \lambda\xi, \tag{5.4.45}$$

where $\lambda \in \mathbb{R}$ is a scalar. It follows that $\mathcal{G}_{\hat{\mu}_e}$ is the subspace spanned by the vector ξ, which, in turn, implies that

$$\mathcal{G}_{\hat{\mu}_e} = \mathcal{G}_{\hat{\xi}}.$$

Recall that in Lemma 3.4.5, V_ξ is invariant along the group orbit, $\mathcal{G}_{\hat{\xi}} \cdot x_e$. From Remark 5.4.11, we conclude that for this case, the function $\tilde{V}_\xi$ defined in (3.4.8) is sufficient for determining stability. We summarize the discussion in the following corollary.

COROLLARY 5.4.12

We consider a gyroscopic system with symmetry (Q, K, Y, V, G).

111

(i) For the case that $Q = G$, positive definiteness of $\mathbf{B}_\xi$ at relative equilibrium, defined in (5.4.34), on $\mathcal{V}_{RIG} \times \mathcal{V}_{RIG}$ implies relative stability modulo G_μ.

(ii) For the case that $G = SO(3)$, a strict local minimizer of the function $\tilde{V}_\xi$, defined in (3.4.8) on the space Q/G_ξ induced by the augmented potential V_ξ corresponds to stable relative equilibrium.

$\blacksquare$

Now we implement the energy-momentum method in more detail for the case of $G = SO(3)$. Through the isomorphism between $\mathbb{R}^3$ and skew symmetric matrices defined in (2.2.1), we could define the *locked inertia dyadic* $\mathbf{I}^\circ_{lock}(x)$ as

$$\langle \hat{\xi}, \, \mathbf{I}_{lock}(x)\hat{\eta} \rangle \;=\; \xi \cdot \mathbf{I}^\circ_{lock}(x)\eta, \tag{5.4.46}$$

where we have used the *trace* pairing, cf. (2.2.4) and the matrices $\frac{1}{2}\mathbf{I}_{lock}(x)$, $\mathbf{I}^\circ_{lock}(x)$ are related by the formula in (2.2.2e). Also, we may define

$$\mathbf{I}_Y(x) \;=\; \widehat{\mathbf{I}^\circ_Y(x)}, \tag{5.4.47}$$

where $\mathbf{I}^\circ_Y(x) \in \mathbb{R}^3$. Namely,

$$\langle \mathbf{I}_Y(x), \, \hat{\eta} \rangle \;=\; \mathbf{I}^\circ_Y(x) \cdot \eta. \tag{5.4.48}$$

With these two objects, we have the following new representations,

$$\tilde{J}^\circ(x, \, \hat{\zeta}) \;=\; \mathbf{I}^\circ_{lock}(x)\zeta \;+\; \mathbf{I}^\circ_Y(x), \tag{5.4.49a}$$

$$V_{\hat{\xi}} \;=\; V(x) \;-\; \frac{1}{2}\xi \cdot \mathbf{I}^\circ_{lock}(x)\xi \;-\; \mathbf{I}^\circ_Y(x) \cdot \xi, \tag{5.4.49b}$$

$$\mathrm{ident}^{Y\circ}_{\hat{\xi}}(x)\delta x \;=\; -(D\mathbf{I}^\circ_{lock}(x)\delta x)\,\xi \;-\; D\mathbf{I}^\circ_Y(x)\delta x, \tag{5.4.49c}$$

The bilinear form defined in (5.4.34) is now

$$\mathbf{B}_{\hat{\xi}}(\delta x_1, \, \delta x_2) \;=\; \mathrm{ident}^{Y\circ}_{\hat{\xi}}(x_e)\delta x_1 \cdot \mathbf{I}^\circ_{lock}(x_e)^{-1}\mathrm{ident}^{Y\circ}_{\hat{\xi}}(x_e)\delta x_2$$
$$+\; D^2 V_\xi(x_e) \cdot \delta x_1 \cdot \delta x_2. \tag{5.4.50}$$

The Arnold block in (5.4.38) can be then written as, cf. (5.4.44),

$$\langle ad^*_{\hat{\eta}}\hat{\mu}_e, \, \mathbf{I}_{lock}(x_e)^{-1}\,ad^*_{\hat{\eta}}\hat{\mu}_e \rangle \;+\; \langle ad^*_{\hat{\eta}}\hat{\mu}_e, \, ad_{\hat{\eta}}\hat{\xi} \rangle$$

$$=\; (\mu_e \times \eta) \cdot \mathbf{I}^\circ_{lock}(x_e)^{-1}(\mu_e \times \eta) \;+\; (\mu_e \times \eta) \cdot (\eta \times \xi) \tag{5.4.51}$$

$$=\; \lambda^2\,(\xi \times \eta) \cdot \left(\mathbf{I}^\circ_{lock}(x_e)^{-1} \;-\; \frac{1}{\lambda}\mathbf{1}\right)(\xi \times \eta)$$

112

It is thus clear that for the Arnold block, we need to check the definiteness of the matrix $\mathbb{I}^o_{lock}(x_e)^{-1} - \frac{1}{\lambda}\mathbb{1}$ along all directions except ξ. Note that here λ is *not* an eigenvalue of the locked inertia dyadic in contrast with the case of simple mechanical systems with symmetry. The gyroscopic field affects λ through the gyro-momentum term, cf. (5.4.45).

Since the examples we shall consider later are all with $G = SO(3)$, the above formulae will be used frequently.

CHAPTER VI

Rigid Body in a Central Gravitational Field

This chapter concerns the dynamics of a rigid body of finite extent moving under the influence of a central gravitational field. The principal motivation behind this work is to reveal the hamiltonian structure of the n-body problem for masses of finite extent and to understand the approximation inherent to modeling the system as the motion of point masses. To this end, explicit account is taken of effects arising because of the finite extent of the moving body. In the spirit of Arnold and Smale, exact models of spin-orbit coupling are formulated, with particular attention given to the underlying Lie group framework. Hamiltonian structures associated with such models are carefully constructed and shown to be *non-canonical*. Special motions, namely relative equilibria, are investigated in detail and the notion of a *non-great circle* relative equilibrium is introduced. Non-great circle motions cannot arise in the point mass model. In our analysis, a variational characterization of relative equilibria is found to be very useful.

The *reduced* hamiltonian formulation in this Chapter suggests a systematic approach to approximation of the underlying dynamics based on series expansion of the reduced Hamiltonian. We will also establish nonlinear stability results for certain families of relative equilibria. Here the energy-Casimir method and the Lagrange multiplier methods are proved to be useful. This Chapter follows closely the discussions in [72].

6.1. Hamiltonian Setting

In the study of the Newtonian (gravitational) many-body problem, it is customary to treat the bodies as point masses. See (Sternberg [68], Smale [64], and Abraham and Marsden [2]). However the proper accounting of stable planetary spins for instance, would seem to require the consideration of bodies of finite extent which will be assumed rigid (possibly nonhomogeneous) as a first approximation. The works of Duboshin

[15], Ermenko [18], Elipe and Cid [16], Elipe and Ferrer [17], are concerned with the existence of special solutions (e.g. central configurations) in the Newtonian many-rigid-body problem. However, in these papers, the natural geometric and group-theoretic underpinnings of the problem are not exploited to the full extent possible. Here we work out the noncanonical hamiltonian structure of the problem of motion of a rigid body in a central gravitational field.

A configuration of the system is depicted in Figure 6.1.1. Let C denote a fixed gravitating body of mass M (with spherical symmetry) that influences the motion of a rigid body $\mathcal{B}$ of mass m. The inertial frame of reference (of the observer) is attached to C and a body frame is fixed on the rigid body $\mathcal{B}$ at its center of mass. A typical material particle $\bar{q}$ in the rigid body is represented by the inertial vector $q = B\bar{q} + r$, where B is an element of $SO(3)$ (independent of the particle) and r is the vector from C to the center of mass of body $\mathcal{B}$. At any instant, the configuration of the rigid body $\mathcal{B}$ is determined uniquely from the pair $(B, r) \in SE(3)$, the special Euclidean group of rigid motions in $\mathbb{R}^3$.
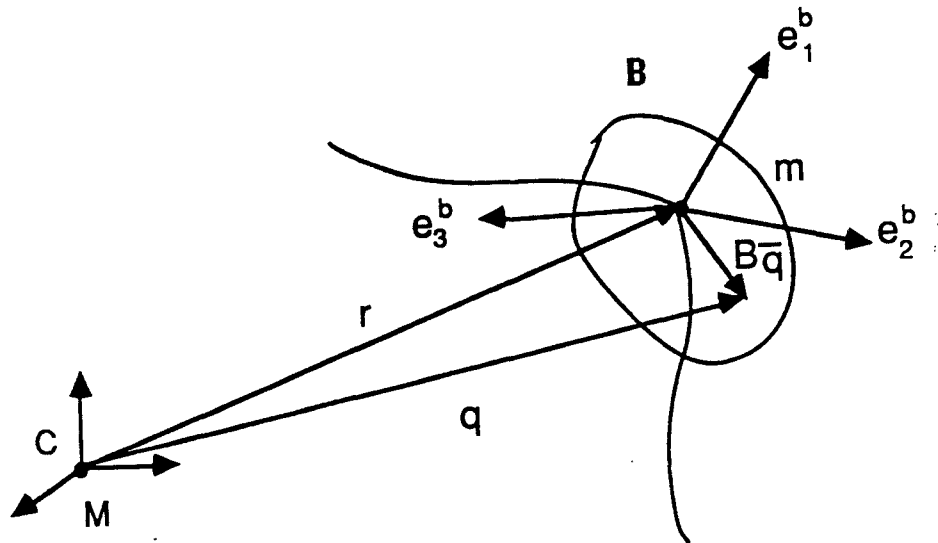

Figure 6.1.1. Rigid Body in a Central Gravitational Field

The kinetic energy of the rigid body relative to the observer at C is,

$$T = \frac{1}{2} \int_{\mathcal{B}} |\dot{q}|^2 \, dm(\bar{\mathbf{q}})$$

where $dm\,(\cdot)$ denotes the mass measure of the body. Here onwards, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. It is an elementary fact that the above expression simplifies to the formula, cf. (3.5.2),

$$T = \frac{1}{2} < \Omega, \, I\Omega > + \frac{m}{2} |\dot{r}|^2 \qquad (6.1.1)$$

where Ω is the body angular velocity vector of the rigid body , m is the total mass of the body and $\mathbf{I}$ is the moment of inertia tensor of $\mathcal{B}$ in the body frame. We note that $K = 2T$ defines a riemannian metric on $SE(3)$, the configuration space.

The gravitational potential energy of the body $\mathcal{B}$ is given by,

$$V = - \int_{\mathcal{B}} \frac{GM}{|q|} \, dm(\bar{\mathbf{q}}) = - \int_{\mathcal{B}} \frac{GM \, dm(\bar{\mathbf{q}})}{|r + B\bar{\mathbf{q}}|}, \qquad (6.1.2)$$

where G is the universal gravitational constant. The Lagrangian for the problem is then a function

$$L : T\,(SE(3)) \to \mathbb{R},$$
$$(B, r, \Omega, \dot{r}) \mapsto T - V. \qquad (6.1.3)$$

The inertial observer at C has the freedom to change his frame of reference to a new orientation. This corresponds to an $SO(3)$ action on the configuration space $\mathcal{C} = SE(3)$:

$$\Phi : SO(3) \times \mathcal{C} \to \mathcal{C}$$
$$(R, (B, r)) \mapsto (RB, Rr). \qquad (6.1.4)$$

It is easily checked that this action leaves invariant the kinetic energy T (riemannian metric on $\mathcal{C}$) and the potential V.

The Hamiltonian $H = T + V$ is given by

$$H = \frac{1}{2} < \Pi, \, \mathrm{I}^{-1}\Pi > + \frac{\mid p \mid^2}{2m} - \int_{\mathcal{B}} \frac{GM}{\mid r + B\bar{q} \mid} dm \, (\bar{q}), \qquad (6.1.5)$$

where $\Pi = \mathrm{I}\,\Omega$ is the body angular momentum of the rigid body $\mathcal{B}$, and $p = m\dot{r}$ is the spatial linear momentum of the body. One has also the formula,

$$\pi = B\mathrm{I}\Omega + r \times m\dot{r}$$

$$= B\Pi + r \times p,$$

for the *spatial angular momentum* of the rigid body.

It can be verified that $\pi = \pi \, (\Pi, B, r, p)$ is an Ad*-equivariant momentum mapping for the lifted action Φ^{T^*} on $T^*SE(3)$ and hence is a conserved quantity for the dynamics X_H . This is further equivalent to Euler's balance law. To see this, let $\mathcal{F}_{\text{resultant}}$ denote the force resultant on the rigid body. Then,

$$\mathcal{F}_{\text{resultant}} = - \int_{\mathcal{B}} \frac{GM(r + B\bar{q})}{\mid r + B\bar{q} \mid^3} \, dm(\bar{q}),$$

and by linear momentum balance,

$$\dot{p} = \mathcal{F}_{\text{resultant}}. \qquad (6.1.6)$$

On the other hand, the torque resultant,

$$\mathcal{T}_{\text{resultant}} = - \int_{\mathcal{B}} \frac{(r + B\bar{q}) \times (r + B\bar{q}) \, GM}{\mid r + B\bar{q} \mid^3} dm(\bar{q})$$

$$= 0.$$

Thus angular momentum (or Euler's) balance law yields:

$$\dot{\pi} = 0. \qquad (6.1.7)$$

Collecting together the balance laws one can write the spatial form of the dynamics as

$$\dot{\pi} = 0,$$

$$\dot{p} = -\int_B \frac{GM\,(r + B\bar{q})}{\mid r + B\bar{q}\mid^3}\,dm(\bar{q}),$$

$$\dot{B} = \hat{\omega}\,B,$$

$$\dot{r} = p/m. \qquad\qquad (6.1.8)$$

where ω is the *spatial angular velocity* of B defined by $\dot{B} = \hat{\omega}B$, and we have the relation $\omega = B\Omega$. Equivalently, in mixed body and space variables (Π, B, r, p) we get,

$$\dot{\Pi} = \Pi \times \mathbf{I}^{-1}\Pi + \int_B \frac{GM(B^T r \times \bar{q})}{\mid r + B\bar{q}\mid^3}\,dm(\bar{q}),$$

$$\dot{p} = -\int_B \frac{GM\,(r + B\bar{q})}{\mid r + B\bar{q}\mid^3}\,dm(\bar{q}),$$

$$\dot{B} = B\widehat{\mathbf{I}^{-1}\Pi},$$

$$\dot{r} = p/m. \qquad\qquad (6.1.9)$$

We remark that, with the Lagrangian (6.1.3), we could apply the techniques developed in Section 2.2 and obtain the same dynamical equations, cf. Section 4.3.

6.2. Symmetry

Since H is $SO(3)$-invariant, one can induce a Hamiltonian $\tilde{H}$ on the quotient $T^*(SE(3))/SO(3)$ and express the dynamics $X_{\tilde{H}}$ in terms of appropriate reduced variables, see Section 3.3. In the present context it is easy to determine the reduced variables. Note that

$$\Phi^{T^*}: \quad SO(3) \times T^*SE(3) \to T^*SE(3) \qquad\qquad (6.2.1)$$

$$(R,(\Pi, B, r, p)) \mapsto (\Pi, RB, Rr, Rp)$$

is the cotangent lift on $T^*SE(3)$ corresponding to the action (6.1.4). A representative for each equivalence class in $T^*SE(3)/SO(3)$ is given by

$$(\Pi,\ 1,\ B^T r,\ B^T p).$$

Thus the reduced variables (or convected variables) are:

Π, the body angular momentum,

$\lambda = B^T r$, the convected radius vector from C, and

$\mu = B^T p$, the convected linear momentum.

In terms of these convected variables, the dynamics $X_{\tilde{H}}$ takes the form

$$\dot{\Pi} = \Pi \times I^{-1}\Pi + \int_{\mathcal{B}} \frac{GM (\lambda \times \bar{q})}{|\lambda + \bar{q}|^3} dm(\bar{q}),$$

$$\dot{\lambda} = \lambda \times I^{-1}\Pi + \frac{\mu}{m},$$ (6.2.2)

$$\dot{\mu} = \mu \times I^{-1} \Pi - \int_{\mathcal{B}} \frac{GM (\lambda + \bar{q})}{|\lambda + \bar{q}|^3} dm(\bar{q}),$$

and the Hamiltonian $\tilde{H}$ is given by,

$$\tilde{H} = \frac{1}{2}\langle \Pi, I^{-1}\Pi \rangle + \frac{|\mu|^2}{2m} - \int_{\mathcal{B}} \frac{GM}{|\lambda + \bar{q}|} dm(\bar{q}).$$ (6.2.3)

Equations (6.2.2) with Hamiltonian (6.2.3) are the Poisson reduced equations on $T^*SE(3)/SO(3) \simeq so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$. The reduced Poisson bracket can be found in the following way. The resulting bracket captures the geometry of the central force field problem.

Let $\langle \cdot, \cdot \rangle$ denote the pairing between $T^*SO(3)$ and $TSO(3)$ as defined in (2.2.5). $P = T^*SE(3)$ carries a canonical symplectic structure and hence a Poisson structure $\{\cdot, \cdot\}_P$ given by

$$\{\bar{f}, \bar{g}\}_P(B, B\hat{\Pi}, r, p) = \langle D_B\bar{f}, \frac{\partial \bar{g}}{\partial B\hat{\Pi}} \rangle - \langle D_B\bar{g}, \frac{\partial \bar{f}}{\partial B\hat{\Pi}} \rangle + \frac{\partial \bar{f}}{\partial r} \cdot \frac{\partial \bar{g}}{\partial p} - \frac{\partial \bar{g}}{\partial r} \cdot \frac{\partial \bar{f}}{\partial p},$$

where $\frac{\partial \bar{f}}{\partial r} \cdot \frac{\partial \bar{g}}{\partial p}$ denotes the natural pairing, i.e. the Euclidean inner product on $\mathbb{R}^3$.

The group $G = SO(3)$ acts on $SE(3)$ by left multiplication, cf. (6.1.4). With reduced variables, Π, λ, μ, we will compute the reduced Poisson structure on $T^*SE(3)/SO(3) \simeq so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$. Since $so^*(3) \simeq \mathbb{R}^3$, the question is equivalent to finding a Poisson structure on $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$.

Let $f, g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$, and define $\bar{f}, \bar{g} \in C^\infty(T^*SE(3))$ as

$$\bar{f}(B, B\hat{\Pi}, r, p) = f(\Pi, B^T r, B^T p).$$

119

By the definition of reduced Poisson structure, we have

$$\{f,g\}_{P/G}(\Pi,\lambda,\mu) = \{\bar{f},\bar{g}\}_P(B,B\hat{\Pi},B\lambda,B\mu).$$

(The right hand side is the canonical bracket in $T^*SE(3)$.) Then, by the canonical bracket on $T^*SE(3)$,

$$\{f,g\}_{P/G}(\Pi,\lambda,\mu) = \langle D_B\bar{f}, \frac{\partial \bar{g}}{\partial B\hat{\Pi}}\rangle - \langle D_B\bar{g}, \frac{\partial \bar{f}}{\partial B\hat{\Pi}}\rangle + \frac{\partial \bar{f}}{\partial r}\cdot\frac{\partial \bar{g}}{\partial p} - \frac{\partial \bar{g}}{\partial r}\cdot\frac{\partial \bar{f}}{\partial p}.$$

Instead of computing each element in the above formula individually, we compute the *differential* of $\bar{f}$. An argument similar to the one in constructing representations for second tangent bundle in Section 2.2 will be applied to find global representations of elements in $TT^*SE(3)$, $T^*T^*SE(3)$. Let $W = (B\hat{v}_1, B(\hat{v}_1\hat{\Pi} + \hat{v}_2), v_3, v_4) \in T_{(B,B\hat{\Pi},r,p)}T^*SE(3)$. It generates the curve

$$\left(Be^{\epsilon\,\hat{v}_1}, Be^{\epsilon\,\hat{v}_1}(\hat{\Pi} + \epsilon\,\hat{v}_2), r + \epsilon\,v_3, p + \epsilon\,v_4\right) \subset T^*SE(3)$$

Thus the differential is given by,

$$df(B,B\hat{\Pi},r,p)\cdot W$$
$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \bar{f}\left(Be^{\epsilon\,\hat{v}_1}, Be^{\epsilon\,\hat{v}_1}(\hat{\Pi} + \epsilon\,\hat{v}_2), r + \epsilon\,v_3, p + \epsilon\,v_4\right)$$
$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f\left(\Pi + \epsilon\,v_2, e^{\epsilon\,\hat{v}_1^T}B^T(r + \epsilon\,v_3), e^{\epsilon\,\hat{v}_1^T}B^T(p + \epsilon\,v_4)\right)$$
$$= \frac{\partial f}{\partial\Pi}\cdot v_2 + \frac{\partial f}{\partial\lambda}\cdot(\hat{v}_1^T B^T r + B^T v_3) + \frac{\partial f}{\partial\mu}\cdot(\hat{v}_1^T B^T p + B^T v_4)$$
$$= \left(-\lambda\times\frac{\partial f}{\partial\lambda} - \mu\times\frac{\partial f}{\partial\mu}\right)\cdot v_1 + \frac{\partial f}{\partial\Pi}\cdot v_2 + B\frac{\partial f}{\partial\lambda}\cdot v_3 + B\frac{\partial f}{\partial\mu}\cdot v_4.$$

Let the elements in $T^*_{(B,B\hat{\Pi},r,p)}T^*SE(3)$ be denoted as, cf. (2.2.6),

$$\left(B(\hat{b}\hat{\Pi} + \hat{a}), B\hat{b}, c, d\right).$$

We have

$$a = -\lambda\times\frac{\partial f}{\partial\lambda} - \mu\times\frac{\partial f}{\partial\mu}, \quad b = \frac{\partial f}{\partial\Pi},$$
$$c = B\frac{\partial f}{\partial\lambda}, \quad d = B\frac{\partial f}{\partial\mu}.$$

120

Thus we obtain

$$D_B \bar{f} = B\left(\widehat{\frac{\partial f}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial f}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial f}{\partial \mu}})\right), \quad D_{B\hat{\Pi}}\bar{f} = B\widehat{\frac{\partial f}{\partial \Pi}},$$

$$\frac{\partial \bar{f}}{\partial r} = B\frac{\partial f}{\partial \lambda}, \qquad \frac{\partial \bar{f}}{\partial p} = B\frac{\partial f}{\partial \mu}.$$

The reduced bracket can then be derived as

$$\{f,g\}_{P/G}(\Pi, \lambda, \mu)$$

$$= \; <B\left(\widehat{\frac{\partial f}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial f}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial f}{\partial \mu}})\right), B\widehat{\frac{\partial f}{\partial \Pi}}> + B\frac{\partial f}{\partial \lambda} \cdot B\frac{\partial g}{\partial \mu}$$

$$<B\left(\widehat{\frac{\partial g}{\partial \Pi}}\hat{\Pi} - (\lambda \times \widehat{\frac{\partial g}{\partial \lambda}}) - (\mu \times \widehat{\frac{\partial g}{\partial \mu}})\right), B\widehat{\frac{\partial f}{\partial \Pi}}> + B\frac{\partial g}{\partial \lambda} \cdot B\frac{\partial f}{\partial \mu},$$

$$= \frac{1}{2}tr\left(\hat{\Pi}^T \widehat{\frac{\partial f}{\partial \Pi}}^T \widehat{\frac{\partial g}{\partial \Pi}}\right) - \frac{1}{2}tr\left(\hat{\Pi}^T \widehat{\frac{\partial g}{\partial \Pi}}^T \widehat{\frac{\partial f}{\partial \Pi}}\right) + \frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu}$$

$$+ \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu}\right) - \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu}\right),$$

$$= - \Pi \cdot (\frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Pi}) + \frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu}$$

$$+ \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu}\right) - \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu}\right)$$

$$= (\frac{\partial f}{\partial \Pi}^T \quad \frac{\partial f}{\partial \lambda}^T \quad \frac{\partial f}{\partial \mu}^T)\begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial g}{\partial \Pi} \\ \frac{\partial g}{\partial \lambda} \\ \frac{\partial g}{\partial \mu} \end{pmatrix}.$$

In terms of the notation introduced before, the matrix form for the Poisson tensor Λ is

$$\begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix}. \tag{6.2.4}$$

REMARK 6.2.1

The reduced Poisson structure derived here is very closely related to the one derived by Krishnaprasad and Marsden for the dynamics of a rigid body with a flexible attachment [38]. The key link is the geometry. In [38] the unreduced phase space is infinite dimensional and is given by

$$P_\infty = T^*SO(3) \times T^*C$$

where $C = \{f : [0, L] \to \mathbb{R}^3 \mid f \text{ is smooth}\}$ is the configuration space for the string attachment. In the present paper the unreduced phase space is

$$P = T^*SE(3) = T^*SO(3) \times T^*\mathbb{R}^3.$$

In both settings, the reduction is by $SO(3)$. In [38], the Poisson bracket takes the form

$$\{f, g\}_{P_\infty/G} = -\Pi \cdot (\frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Pi}) + \int_0^L \left(\frac{\partial f}{\partial \lambda} \cdot \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \lambda} \cdot \frac{\partial f}{\partial \mu} \right) ds$$
$$+ \int_0^L \frac{\partial f}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial g}{\partial \lambda} + \mu \times \frac{\partial g}{\partial \mu} \right) ds - \int_0^L \frac{\partial g}{\partial \Pi} \cdot \left(\lambda \times \frac{\partial f}{\partial \lambda} + \mu \times \frac{\partial f}{\partial \mu} \right) ds,$$

where the convected variables λ, μ are $\mathbb{R}^3$ valued functions on $[0, L]$. As we let the flexible attachment become vanishingly small ($L \to 0$) with infinitely large density, $\{\cdot, \cdot\}_{P_\infty/G}$ "collapses" to $\{\cdot, \cdot\}_{P/G}$.

∎

In terms of the Poisson tensor Λ on $so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, (6.2.4), equations (6.2.2) take the compact form,

$$\begin{pmatrix} \dot{\Pi} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \hat{\Pi} & \hat{\lambda} & \hat{\mu} \\ \hat{\lambda} & 0 & I \\ \hat{\mu} & -I & 0 \end{pmatrix} \begin{pmatrix} \nabla_\Pi \tilde{H} \\ \nabla_\lambda \tilde{H} \\ \nabla_\mu \tilde{H} \end{pmatrix} = \Lambda \nabla \tilde{H}. \qquad (6.2.5)$$

The Poisson structure is rank-degenerate, and there are nontrivial Casimir functions of Π, λ, μ. Casimir functions are *kinematic* conserved quantities for equations of the form (6.2.5). In fact, any function C_ϕ of the form, cf. (5.3.7),

$$C_\phi = \phi (|\Pi + \lambda \times \mu|^2),$$

is a Casimir function. Here $\phi : \mathbb{R} \to \mathbb{R}$ is any smooth scalar function. Moreover, these are the only Casimir functions defined on the open set of generic points of Λ.

From the general properties of Casimir functions, we know that C_ϕ is an integral invariant for any Hamiltonian vector field and in particular for $X_{\tilde{H}}$. It is further important to note that replacing $\tilde{H}$ by a suitable approximation (such as derived from series expansions of the Newtonian potential term) does not affect the integral invariance of C_ϕ. This is of some use in developing an analytic perturbation theory.

6.3. Relative Equilibrium

With the symmetry discussed in the previous section, we put our system into the framework of simple mechanical systems with symmetry. The elements are,

$$Q = SE(3),$$

$$K\left((U_1, U_2), (W_1, W_2)\right) = tr(U_1 I W_1^T) + m < U_2, W_2 >_E,$$

$$V(B, r) = -\int_B \frac{GM}{|r + B\bar{q}|} dm(\bar{q}),$$

$$G = SO(3),$$

(6.3.1)

where (U_1, U_2), $(W_1, W_2) \in T_{(B, r)}SE(3)$, and I is the coefficient of inertia of the rigid body. The superscript T in W_i denotes matrix transpose.

With this framework, we now discuss the concept of relative equilibrium defined in Definition 3.4.2. For the dynamics $X_{\tilde{H}}$ of a rigid body in a central gravitational field, the relative equilibria are determined by setting the time derivatives in equation (6.2.2) (or (6.2.5)) to zero. On the other hand, in general position, i.e. $\Pi \neq 0$, ∇C_ϕ spans the kernel of Λ. Thus we have the energy-Casimir characterization of relative equilibria in general position: (Π, λ, μ) is a relative equilibrium *iff*

$$\nabla \tilde{H} = \nabla C_\phi, \qquad \text{for suitable } \phi, \qquad (EC)$$

iff (Lagrange multiplier characterization)

$$\begin{pmatrix} I^{-1}\Pi \\ \nabla_\lambda \tilde{V} \\ \frac{\mu}{m} \end{pmatrix} = \alpha \begin{pmatrix} \Pi + \lambda \times \mu \\ \mu \times (\Pi + \lambda \times \mu) \\ (\Pi + \lambda \times \mu) \times \lambda \end{pmatrix}. \qquad (LM)$$

where $\alpha \neq 0$ is a constant and

$$\tilde{V}(\lambda) = -\int_B \frac{GM}{|\lambda + \bar{q}|} dm(\bar{q}).$$

On the other hand, the algorithm in the principle of symmetric criticality provides us an alternate characterization of relative equilibria, cf. Algorithm 3.4.4, or 3.4.7. In what follows we apply this principle to find the relative equilibria for the problem of rigid body motion in a central force field.

For $\hat{\xi} \in so(3)$, the corresponding infinitesimal generator of the group action on Q can be found as

$$\xi_M(B, r) = (\hat{\xi}B, \hat{\xi}r). \qquad (6.3.2)$$

We then have

$$K(\xi_M(B, r), \xi_M(B, r)) = \; < B^T\xi, \; IB^T\xi >_E \; + \; m \mid \xi \times r \mid^2, \qquad (6.3.3)$$

and

$$V_\xi(B, r) = -\frac{1}{2} < B^T\xi, \; I \; B^T\xi >_E \; - \frac{1}{2} \; m \mid \xi \times r \mid^2$$
$$- \int_B \frac{GM}{\mid r + B\bar{\mathbf{q}} \mid} \; dm(\bar{\mathbf{q}}). \qquad (6.3.4)$$

We then get the first-order conditions for (B, r) to be a critical point:

$$(i) \qquad m \; \xi \times (\xi \times r) \; + \; \int_B \frac{(r + B\bar{\mathbf{q}}) \, GM}{\mid r + B\bar{\mathbf{q}} \mid^3} dm(\bar{\mathbf{q}}) \; = \; 0$$

$$(6.3.5)$$

$$(ii) \qquad \xi \times (BIB^T\xi) \; - \; \int_B \frac{(r \times B\bar{\mathbf{q}}) \, GM}{\mid r + B\bar{\mathbf{q}} \mid^3} \; dm(\bar{\mathbf{q}}) \; = \; 0.$$

Next, we calculate p_e in Step 2. The map $K^\flat$ can be found as follows. For $(\hat{w}_1 B, w_2)$, $(\hat{u}_1 B, u_2) \in T_{(B,r)}SE(3)$,

$$K^\flat \; (\hat{w}_1 B, \; w_2) \; (\hat{u}_1 B, \; u_2)$$

$$= \; tr(\hat{w}_1 B \; IB^T \hat{u}_1^T) \; + \; m \; < \; w_2, \; u_2 >_E$$

$$= \; < u_1, \; BIB^T w_1 >_E \; + \; < u_2, \; mw_2 >_E \; .$$

Thus

$$K^\flat(\hat{w}_1 B, \; w_2) = ((\widehat{BIB^T w_1}) \, B, \; mw_2) \; \in \; T^*_{(B,r)} \; SE(3).$$

We then have

$$p_e = K^\flat(\xi_M(q_e)) = \left((\widehat{BIB^T\xi}) \, B, \; m\hat{\xi}r \right)$$

$$= \left(B(\widehat{IB^T\xi}), \; m\hat{\xi} \, r \right). \qquad (6.3.6)$$

Note that in the formula for p_e, the two components correspond to the angular momentum and linear momentum respectively. If we let μ denote the body representation of the linear momentum, we get

124

$$(iii) \quad \mu = mB^T \hat{\xi} r.$$

Substituting $\Omega = B^T \xi$, $\lambda = B^T r$, conditions (i), (ii), (iii) read

$$(i') \quad m\, \Omega \times (\Omega \times \lambda) + \int_B \frac{(\lambda + \bar{\mathbf{q}})GM}{|\lambda + \bar{\mathbf{q}}|^3}\, dm(\bar{\mathbf{q}}) = 0,$$

$$(ii') \quad \Omega \times I\Omega - \int_B \frac{(\lambda \times \bar{\mathbf{q}})\, GM}{|\lambda + \bar{\mathbf{q}}|^3}\, dm(\bar{\mathbf{q}}) = 0, \tag{6.3.7}$$

and

$$(iii') \quad \mu = m\,(\Omega \times \lambda).$$

These conditions are identical to the conditions obtained from the reduced dynamics (6.2.2) and the definition of relative equilibrium.

Now, if we take the cross product with λ on both sides of (i'), we get

$$m\lambda_e \times (\Omega_e \times (\Omega_e \times \lambda_e)) + \int_B \frac{(\lambda_e \times \bar{\mathbf{q}})\, GM}{|\lambda_e + \bar{\mathbf{q}}|^3}\, dm(\bar{\mathbf{q}}) = 0.$$

(Here again the subscripts e refer to equilibrium.) Comparing it with (ii'), we obtain

$$m\, \lambda_e \times (\Omega_e \times (\Omega_e \times \lambda_e)) + \Omega_e \times I\Omega_e = 0.$$

By standard identities in vector analysis, we get

$$\Omega_e \times (\mathbf{I} - m\lambda_e\lambda_e^T)\, \Omega_e = 0. \tag{6.3.8}$$

We conclude that Ω_e must be an eigenvector of the matrix $\mathbf{I} - m\lambda_e\lambda_e^T$.

Let k_e denote the corresponding eigenvalue. Then one can obtain the relative equilibrium characterization (LM) from (6.3.7) by setting,

$$\alpha = \frac{1}{k_e + m\,|\lambda_e|^2} \tag{6.3.9}$$

Conversely, using the identity,

$$\lambda \times ((\Omega \times \lambda) \times \Omega) = -\Omega \times (\lambda \times (\Omega \times \lambda))$$
$$= (\lambda \cdot \Omega)\mu,$$

and a few further algebraic manipulations, one can derive (6.3.7) from the relative equilibrium characterization (LM). We leave the verification to the reader. Thus the two characterizations are equivalent. Of course, for simple mechanical systems with symmetry, the equivalence of the characterization (EC) or (LM) and the variational characterization based on the augmented potential V_ξ holds in general.

Note that we fix ξ while searching for critical points of V_ξ. Thus $\Omega = B^T \xi$ is of fixed norm as B varies over SO(3).

Let

$$| \Omega |^2 = | \xi |^2 = \beta.$$

Define $\tilde{V}_\beta (\Omega, \lambda)$ to be $V_\xi(B, \gamma)$ expressed in the convected variables Ω, λ. Then,

$$\tilde{V}_\beta (\Omega, \lambda) = -\frac{1}{2} <\Omega, I\Omega>_E - \frac{m}{2} | \Omega \times \lambda |^2$$
$$+ \hat{V} (\lambda). \tag{6.3.10}$$

Clearly, the critical points of $\tilde{V}_\beta$ on the sphere $| \Omega |^2 = \beta$ satisfy the unconstrained variational principle,

$$d(\tilde{V}_\beta + \frac{1}{2\alpha} | \Omega |^2) = 0, \tag{6.3.11}$$

where $1/2c$ is a Lagrange multiplier. The first-order conditions associated to (6.3.11) are,

$$I \Omega + m \lambda \times (\Omega \times \lambda) = \frac{1}{\alpha} \Omega \tag{6.3.12a}$$
$$m (\Omega \times \lambda) \times \Omega = \nabla_\lambda \hat{V}. \tag{6.3.12b}$$

126

These are exactly the equations we get by eliminating $\mu = m(\Omega \times \lambda)$ in the relative equilibrium characterization (LM). The unconstrained variational principle (6.3.11), parametrized by α, and the associated first-order conditions (6.3.12) appear to be most suited to the explicit computation of relative equilibria. Before we proceed with such specific computations we make some general geometric observations concerning relative equilibria.

Observe that, by taking the inner product of both sides of (iii') with λ_e, we get

$$< \lambda_e, \ \mu_e >_E \ = \ 0 \qquad (6.3.13)$$

at a relative equilibrium (λ_e, μ_e, Ω_e). If (r_e, B_e) is a relative equilibrium configuration, then the dynamical motion is such that

$$r(t) \ = \ e^{t\hat{\xi}} \, r_e$$
$$B(t) \ = \ e^{t\hat{\xi}} \, B_e. \qquad (6.3.14)$$

This follows from (3.4.1) that at a relative equilibrium the dynamical orbit is just a group orbit.

PROPOSITION 6.3.1

In relative equilibrium, the radius vector $r(t)$ generates a right circular cone.

Proof

From (6.3.14),

$$< r(t), \ r(t) >_E \ = \ < \ e^{t\hat{\xi}} \, r_e, \ e^{t\hat{\xi}} \, r_e \ >_E \ = \ < \ r_e, \ r_e \ >_E \ .$$

Also

$$< \ r - \frac{< r, \xi >_E \ \xi}{|\xi|^2}, \ r - \frac{< r, \xi >_E \ \xi}{|\xi|^2} \ >_E$$
$$= \ < \ r, \ r \ >_E \ - \ \frac{< r, \xi >_E^2}{|\xi|^2}$$
$$= \ < \ r_e, \ r_e \ >_E \ - \ \frac{< \ e^{t\hat{\xi}} \, r_e, \ e^{t\hat{\xi}} \, \xi >_E^2}{|\xi|^2}$$
$$= \ < \ r_e, \ r_e \ >_E \ - \ \frac{< \ r_e, \ \xi >_E^2}{|\xi|^2}$$
$$= \ \text{constant}.$$
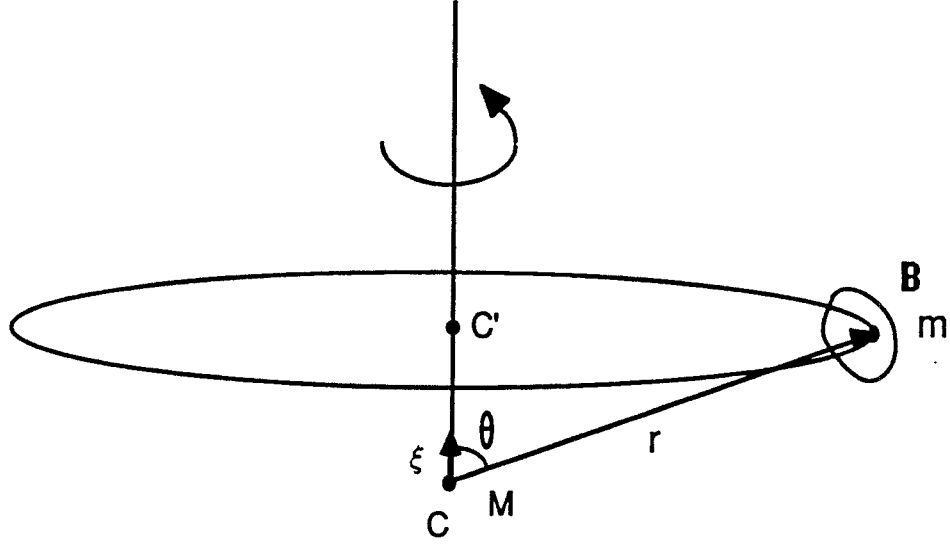
Figure 6.3.1. Cone generated by $r(t)$.

Thus $r(t)$ is a circle of radius $\left(<r_e, r_e>_E - \frac{<r_e, \xi>_E^2}{|\xi|^2} \right)^{\frac{1}{2}}$ centered at $C' = \frac{<r, \xi>_E}{|\xi|^2}\xi$. See Figure 6.3.1.

∎

Next we discuss the notion of *non-great circle motions*. For a rigid body of finite extent, *if* the center of (relative equilibrium) rotation C' *does not* coincide with the center C of the force field, then the stationary motion will be called a non-great circle motion. The existence of such motions is in question. See, e.g. the model problem below and also the gyrostat example in Rumyantsev [59].

From equation (6.1.9),

$$- \int_B \frac{GM\,(r\ +\ B\bar{q})}{|\,r\ +\ B\bar{q}\,|^3} dm(\bar{q})\ =\ \dot{p}\ =\ m\,\ddot{r}$$

$$=\ m\frac{d^2}{dt^2}\left(\ e^{t\hat{\xi}}\,r_e\right)\ =\ m\,e^{t\hat{\xi}}\,\hat{\xi}^2\,r_e.$$

Substituting $r = e^{t\hat{\xi}}\,r_e$ and $B = e^{t\hat{\xi}}\,B_e$ on the left hand side, we get,

$$- \int_B \frac{GM\,(r_e\ +\ B_e\,\bar{q})}{|\,r_e\ +\ B_e\bar{q}\,|^3} dm(\bar{q})\ =\ m\,\hat{\xi}^2\,r_e.$$

128

Taking the inner product of both sides with ξ, we get,

$$- \int_{\mathcal{B}} GM \, \frac{<\xi, \, r_e >_E \; + \; <\xi, \, B_e \, \bar{q} >_E}{|\, r_e \; + \; B_e \bar{q} \,|^3} dm(\bar{q}) \; = \; m < \xi, \, \hat{\xi}^2 \, r_e > \; = \; 0.$$

Hence

$$< \xi, \, r_e >_E \int_{\mathcal{B}} \frac{dm(\bar{q})}{|\, r_e \; + \; B_e \bar{q} \,|^3} \; = \; - \int_{\mathcal{B}} \frac{<\xi, \, B_e \, \bar{q} >_E}{|\, r_e \; + \; B_e \bar{q} \,|^3} dm(\bar{q}).$$

Equivalently,

$$< \xi, \, r_e >_E \; = \; - \int_{\mathcal{B}} \frac{<\xi, \, B_e \, \bar{q} >_E}{|\, r_e \; + \; B_e \bar{q} \,|^3} dm(\bar{q}) \; / \; \int_{\mathcal{B}} \frac{dm(\bar{q})}{|\, r_e \; + \; B_e \bar{q} \,|^3}.$$

The quantity $<\xi, \, r_e >_E$ is proportional to the $cos(\theta)$ (refer to Figure 6.3.1), and C and C' coincide iff $<\xi, \, r_e >_E = 0$. If the body $\mathcal{B}$ were a point mass, $Q = 0$ and hence $<\xi, \, r_e >_E = 0$. If for a rigid body of finite extent, the integral

$$\int_{\mathcal{B}} \frac{<\xi, \, B_e \, \bar{q} >_E}{|\, r_e \; + \; B_e \bar{q} \,|^3} dm(\bar{q}) \neq \; 0,$$

then C, C' are not coincident.

Since

$$< \xi, \, r_e >_E \; = \; < B_e^T \xi, \, B_e^T \, r_e >_E$$

$$= \; \Omega_e \cdot \lambda_e,$$

we conclude that a relative equilibrium $(\lambda_e, \, \Omega_e, \, \mu_e)$ determines a non-great circle motion iff

$$\Omega_e \cdot \lambda_e \neq \; 0. \tag{6.3.15}$$

One can test the non-vanishing condition (6.3.15) in various settings. We now demonstrate that there are examples which do not admit great circle relative equilibria. We first assume that the relative equilibrium is a great circle. Then the equilibrium can be found by solving (from (6.3.12) and $\Omega \cdot \lambda = 0$),

$$\Omega \times I\Omega \; = \; 0,$$

$$\int_{\mathcal{B}} \frac{GM(\lambda + \bar{q})}{|\lambda + \bar{q}|^3} \, dm(\bar{q}) \; = \; m|\Omega|^2 \lambda. \tag{6.3.16}$$

129
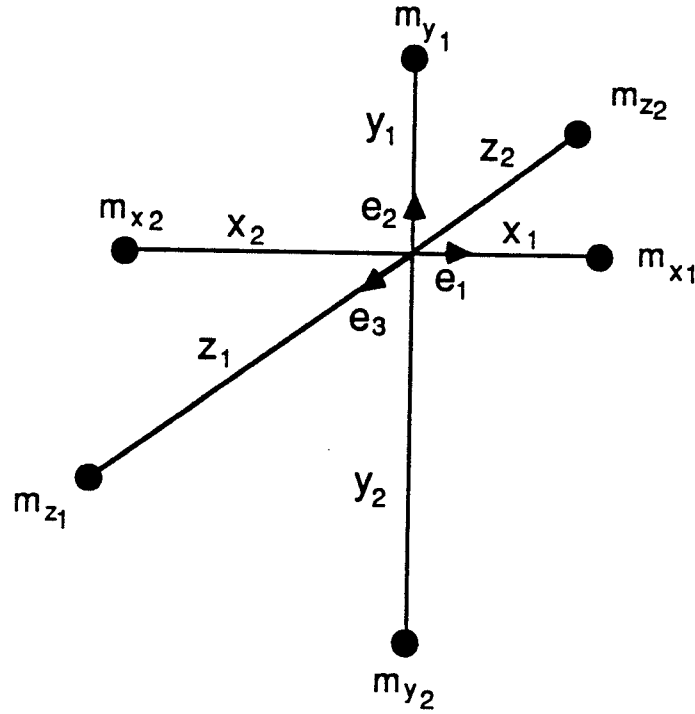
Figure 6.3.2. The "molecule"

We note that given the norm of Ω, the two equations above are decoupled and are equivalent to

1. Ω is an eigenvector of $\mathbf{I}$.

2. λ is a critical point of the function

$$\tilde{V} = \int_B \frac{GM}{|\lambda + \bar{q}|} \, dm(\bar{q}) + \frac{m}{2}|\Omega|^2|\lambda|^2.$$

Moreover, the second condition is equivalent to finding the critical points of

$$\check{V} = \int_B \frac{GM}{|\lambda + \bar{q}|} \, dm(\bar{q}), \qquad (6.3.17a)$$

subject to

$$\frac{1}{2}|\lambda|^2 = \text{constant.} \qquad (6.3.17b)$$

with $m|\Omega|^2$ being the Lagrange multiplier.

Now we consider a model problem. The body is an asymmetric "molecule" consisting of six point masses, two on each principal axis. See Figure 6.3.2.

130

In this example, we know that $\mathbf{I}$ is diagonal and thus for a great circle relative equilibrium, Ω must be along a principal axis. With the notations in Figure 6.3.2, we have the following conditions,

$$x_1 \neq x_2, \qquad x_1 m_{x_1} = x_2 m_{x_2}, \tag{6.3.18a}$$

$$y_1 \neq y_2, \qquad y_1 m_{y_1} = y_2 m_{y_2}, \tag{6.3.18b}$$

$$z_1 \neq z_2, \qquad z_1 m_{z_1} = z_2 m_{z_2}, \tag{6.3.18c}$$

i.e., the body is asymmetric and the coordinate system is assumed to be at the center of mass of the body. We now study the solutions of (6.3.17), namely, the critical point of $\check{V}$ subject to the condition (6.3.17b). Let $|\lambda| = \alpha$, and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. For the molecule, the function $\check{V}$ can be written as,

$$\check{V} = \frac{m_{x_1}}{(\alpha^2 - 2\lambda_1 x_1 + x_1^2)^{\frac{1}{2}}} + \frac{m_{x_2}}{(\alpha^2 + 2\lambda_1 x_2 + x_2^2)^{\frac{1}{2}}} + \frac{m_{y_1}}{(\alpha^2 - 2\lambda_2 y_1 + y_1^2)^{\frac{1}{2}}}$$
$$+ \frac{m_{y_2}}{(\alpha^2 + 2\lambda_2 y_2 + y_2^2)^{\frac{1}{2}}} + \frac{m_{z_1}}{(\alpha^2 - 2\lambda_3 z_1 + z_1^2)^{\frac{1}{2}}} + \frac{m_{z_2}}{(\alpha^2 + 2\lambda_3 z_2 + z_2^2)^{\frac{1}{2}}} \tag{6.3.19}$$

Now we parametrize the vector λ as

$$\lambda_1 = \alpha \cos\phi \cos\theta, \quad \lambda_2 = \alpha \cos\phi \sin\theta, \quad \lambda_3 = \alpha \sin\phi, \tag{6.3.20}$$

with

$$0 \leq \theta < 2\pi, \quad 0 \leq \phi < \pi.$$

This parametrization is valid for every point on the sphere except the intersections of the sphere with the e_3 axis. By substituting (6.3.20) into (6.3.19), the partial derivative of $\check{V}$ with respect to ϕ evaluated at $\phi = 0$ can be found to be

$$\left.\frac{\partial \check{V}}{\partial \phi}\right|_{\phi=0} = \frac{-2\alpha z_1 m_{z_1}}{(\alpha^2 + z_1^2)^{\frac{3}{2}}} + \frac{2\alpha z_2 m_{z_2}}{(\alpha^2 + z_2^2)^{\frac{3}{2}}}$$
$$= 2\alpha z_1 m_{z_1} \left(\frac{-1}{(\alpha^2 + z_1^2)^{\frac{3}{2}}} + \frac{1}{(\alpha^2 + z_2^2)^{\frac{3}{2}}} \right). \tag{6.3.21}$$

Because of (6.3.18c), this will never be zero. Thus the solution of (6.3.17) can never appear when $\phi = 0$, or on the e_1-e_2 plane. Similarly, be choosing other parametrizations, cf. (6.3.20),

131

$$\lambda_1 = \alpha \cos\phi \sin\theta, \quad \lambda_2 = \alpha \sin\phi, \quad \lambda_3 = \alpha \cos\phi \cos\theta,$$

and

$$\lambda_1 = \alpha \sin\phi, \quad \lambda_2 = \alpha \cos\phi \cos\theta, \quad \lambda_3 = \alpha \cos\phi \sin\theta,$$

with the condition (6.3.18), we could show that the solution of (6.3.17) cannot lie on e_1-e_3, e_2-e_3 planes respectively. We have thus established the fact that for the molecule system the solution of (6.3.17) cannot be perpendicular to any principal axis. It follows that $\lambda \cdot \Omega \neq 0$, which contradicts our assumption of a great circle relative equilibrium. Accordingly, we conclude that for this example, there are no great circle relative equilibria.

6.4. Approximations

For typical applications in the modeling of planets or artificial earth satellites, the nominal radius of the orbital motion is very large compared to the dimensions of the orbiting body. Accordingly, it seems appropriate to consider various approximations of the gravitational potential based on Taylor series in a neighborhood of $| \lambda | = \infty$ or equivalently $| r | = \infty$. While such approximations are common in the literature, it is unclear whether the symmetries and conservation laws inherent in the problem are respected by the approximation process.

In the present paper, we take the Poisson reduced model (6.2.5) as the logical starting point for approximations. The Hamiltonian $\bar{H}$ is approximated to various orders of $\epsilon =$ (nominal dimension of body) / (orbital radius), by the Taylor series expansion of the $\bar{V}(\lambda)$ potential term appearing in $\bar{H}$:

$$\bar{V}(\lambda)$$
$$= -\int_B \frac{GM}{|\lambda + \bar{q}|} dm(\bar{q})$$

132

$$= -\frac{GM}{|\lambda|} \int_B dm(\bar{q}) \left\{ 1 - \frac{<Q,\lambda>}{|\lambda|^2} - \frac{1}{2}\frac{|\bar{q}|^2}{|\lambda|^2} + \frac{3}{2}\frac{<Q,\lambda>^2}{|\lambda|^4} + o(|\lambda|^{-4}) \right\}$$

$$= \left[-\frac{GMm}{|\lambda|} \right] + \left[-\frac{GM}{2|\lambda|^3}tr(\mathbf{I}) + \frac{3}{2}\frac{GM}{|\lambda|^5} <\lambda, \mathbf{I}\lambda> \right] + o(|\lambda|^{-5}) \qquad (6.4.1)$$

In (6.4.1) the first term in brackets is of the order ϵ^0 and the next term is of the order ϵ^2. The ϵ^1 term is absent due to the vanishing of $\int_B \bar{q}\, dm(\bar{q})$.

We will therefore consider two approximate *model Hamiltonians*,

$$\tilde{H}_0 = \frac{1}{2} <\Pi, \mathbf{I}^{-1}\Pi> +\frac{|\mu|^2}{2m} - \frac{GMm}{|\lambda|}, \qquad (6.4.2)$$

and

$$\tilde{H}_2 = \frac{1}{2} <\Pi, \mathbf{I}^{-1}\Pi> +\frac{|\mu|^2}{2m} - \frac{GMm}{|\lambda|}$$
$$- \frac{GM}{2|\lambda|^3}tr(\mathbf{I}) + \frac{3}{2}\frac{GM}{|\lambda|^5} <\lambda, \mathbf{I}\lambda> . \qquad (6.4.3)$$

Upon substituting $\tilde{H}_0$ and $\tilde{H}_2$ respectively for $\tilde{H}$ in the Poisson reduced dynamics (6.2.5), one obtains the *order zero* reduced dynamics;

$$\dot{\Pi} = \Pi \times \mathbf{I}^{-1}\Pi,$$
$$\dot{\lambda} = \lambda \times \mathbf{I}^{-1}\Pi + \mu/m, \qquad (6.4.4)$$
$$\dot{\mu} = \mu \times \mathbf{I}^{-1}\Pi - \frac{GMm}{|\lambda|^3} \lambda,$$

and the *order two* reduced dynamics;

$$\dot{\Pi} = \Pi \times \mathbf{I}^{-1}\Pi + \frac{3GM}{|\lambda|^5} \lambda\, \mathbf{I}\, \lambda,$$
$$\dot{\lambda} = \lambda \times \mathbf{I}^{-1}\Pi + \mu/m,$$
$$\dot{\mu} = \mu \times \mathbf{I}^{-1}\Pi - \frac{GMm}{|\lambda|^3} \lambda - \frac{3GM}{2|\lambda|^5} tr(\mathbf{I})\lambda \qquad (6.4.5)$$
$$- \frac{3GM}{|\lambda|^5} \mathbf{I}\lambda + \frac{15}{2}\frac{GM}{|\lambda|^7} <\lambda, \mathbf{I}\lambda> \lambda.$$

As already noted at the end of section 6.2, all such approximations admit a common set of conserved quantities (Casimir functions) of the form $C_\phi = \phi(|\Pi + \lambda \times \mu|^2)$. Since the order 0 dynamics is essentially decoupled, it has additional conserved quantities of the form $\psi(|\Pi|^2)$, and the spin energy $1/2 <\Pi, \mathbf{I}^{-1}\Pi>$. If the body is spherically symmetric, i.e., $\mathbf{I} = k\mathbf{1}$, then the *order two* approximation collapses to the *order zero*

approximation. In general, the *order two* approximation displays nontrivial spin-orbit coupling.

With the order zero approximation of $\tilde{H}$ (6.4.2), the relative equilibria (Ω_e, λ_e) satisfy, from (6.4.4),

$$I\,\Omega_e = k\,\Omega_e \tag{6.4.6a}$$

$$\frac{GMm}{|\lambda|^3}\,\lambda_e = m(\Omega_e \times \lambda_e) \times \Omega_e. \tag{6.4.6b}$$

By taking the inner product of both sides of (6.4.6b) with Ω_e, we conclude that $\lambda_e \cdot \Omega_e = 0$, i.e. all relative equilibria in the *order zero* approximation give rise to great-circle orbits. From (6.4.6b) and the condition $\Omega_e \cdot \lambda_e = 0$, we get the Kepler frequency formula,

$$|\Omega| = \left(\frac{GM}{|\lambda|^3}\right)^{1/2}. \tag{6.4.7}$$

Summarizing, the only relative equilibria for the *order zero* approximation are

(a) Ω_e is a principal axis of I;

(b) λ_e is a vector perpendicular to Ω_e satisfying the Kepler formula (6.4.7);

(c) $\mu_e = m\,(\Omega_e \times \lambda_e)$ completes a triad. $\hspace{2cm}$ (6.4.8)

With the same assumptions as (6.4.8a) and (6.4.8b) above, it is possible verify the existence of "uniformly spinning solutions" to the *order zero* reduced dynamics:

$$\Pi(t) = I\Omega(t) \equiv I\Omega_e$$

$$\lambda(t) = exp\left(t\frac{\omega}{|\Omega_e|}\,\hat{\Omega}_e\right)\lambda_e \tag{6.4.9}$$

$$\mu(t) = m\left(1 + \frac{\omega}{|\Omega_e|}\right)exp\left(t\frac{\omega}{|\Omega_e|}\,\hat{\Omega}_e\right)\hat{\Omega}_e\,\lambda_e,$$

with the modified Kepler frequency formula,

$$(\omega + |\Omega_e|) = \left(\frac{GM}{|\lambda_e|^3}\right)^{1/2}. \tag{6.4.10}$$

134

The quantity ω measures the body spin relative to a moving Frenet-Serret frame at the center of mass of the body.

For the order two model (6.4.3), (6.4.5), the first-order conditions for the variational principle (6.3.11) take the form

$$(\mathbf{I} - m\,\lambda\lambda^T)\Omega = (\frac{1}{\alpha} - m\mid\lambda\mid^2)\,\Omega \qquad (6.4.11)$$

$$m\mid\Omega\mid^2\lambda - m\,(\Omega\cdot\lambda)\,\Omega = \frac{GMm}{\mid\lambda\mid^3}\,\lambda + \frac{3GM}{2\mid\lambda\mid^5}\,tr(\mathbf{I})\lambda$$
$$+\ \frac{3GM}{\mid\lambda\mid^5}\,\mathbf{I}\lambda - \frac{15}{2}\frac{GM}{\mid\lambda\mid^7}\,(\lambda^T\,\mathbf{I}\,\lambda)\lambda$$

The equations (6.4.11) admit a family of solutions (relative equilibria) corresponding to great circle motions:

(a) Ω_e is a principal axis (eigenvector) of $\mathbf{I}$ with corresponding principal moment of inertia I_i, $i = 1, 2, 3$;

(b) λ_e is a principal axis (eigenvector) of $\mathbf{I}$ perpendicular to Ω_e, with associated principal moment of inertia I_j;

(c) $\mu_e = m\,(\Omega_e \times \lambda_e)$;

and, the following modified Kepler frequency formula holds:

$$\mid\Omega_e\mid = \left(\frac{GM}{\mid\lambda_e\mid^3}\right)^{1/2}\left\{1 + \frac{3\,(I_i - 2I_j + I_k)}{2m\mid\lambda_e\mid^2}\right\}^{1/2}. \qquad (6.4.12)$$

In the above relation i, j, k are distinct and take values in $\{1, 2, 3\}$. Hence the correction term in (6.4.12) may be of either sign. It follows that for the *order two* approximation there are twenty-four 1 parameter families of relative equilibria (accounting for Ω being in each of the six directions along the principal axes (with sign) and four directions for λ corresponding to each choice of Ω), the scalar parameter being $\beta = \mid\Omega\mid^2 = \mid\xi\mid^2$ as in Section 5.

This conclusion appears to be a classical result exhibited in different form. See for instance the book of Beletskii [8]. However, the hamiltonian point of view together with

135

the approach of reduction has entirely eliminated the formidable mess of Euler angles and such.

In the following, we show that for practical parameter ranges, all the relative equilibria in the *order two* approximate model are great circle motions. Let

$$(\mathbf{I} - m\lambda\lambda^T)\Omega \;=\; \alpha\Omega,$$

or

$$\mathbf{I}\,\Omega \;-\; \alpha\Omega \;=\; m\lambda\lambda^T\Omega.$$

With the notation $\tau = \lambda^T\Omega$, we have

$$m\tau\lambda \;=\; \mathbf{I}\Omega \;-\; \alpha\Omega. \tag{6.4.13}$$

We note that $\tau \neq 0$ corresponds to solutions that are not great circles, while $\tau = 0$ implies a standard eigenvalue problem. The dot product of (6.4.13) with Ω then yields

$$\alpha \;=\; \frac{1}{|\Omega|^2}(\Omega^T\mathbf{I}\Omega \;-\; m\tau^2),$$

and substitution in (6.4.13) gives

$$\tau\lambda \;=\; \frac{1}{m|\Omega|^2}(|\Omega|^2 - \Omega\Omega^T)\mathbf{I}\Omega \;+\; \frac{1}{|\Omega|^2}\tau^2\Omega. \tag{6.4.14}$$

Taking the dot product with Ω of the second equation in (6.4.11), we get the following equation

$$\left(m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda\right)\tau \;+\; \frac{3}{|\lambda|^2}\Omega^T\mathbf{I}\lambda \;=\; 0. \tag{6.4.15}$$

Assuming $\tau \neq 0$, and multiplying (6.4.15) by τ, we have

$$(m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda)\tau^2 \; + \; \frac{3}{|\lambda|^2}\Omega^T\mathbf{I}\tau\lambda \; = \; 0.$$

With the expression for $\tau\lambda$ in (6.4.14), we obtain the equality,

$$
\begin{aligned}
(m + &\frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda + \frac{3}{|\lambda|^2|\Omega|^2}\Omega^T\mathbf{I}\Omega)\tau^2 \\
&= -\frac{3}{m|\lambda|^2|\Omega|^2}\left\{|\Omega|^2|\mathbf{I}\Omega|^2 - |\Omega^T\mathbf{I}\Omega|^2\right\}.
\end{aligned}
\tag{6.4.16}
$$

But we know that

$$|\Omega|^2|\mathbf{I}\Omega|^2 - |\Omega^T\mathbf{I}\Omega|^2 \; \geq \; 0.$$

Thus (6.4.16) can have a solution with $\tau \neq 0$ only if

$$m + \frac{3}{2|\lambda|^2}tr\mathbf{I} - \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda + \frac{3}{|\lambda|^2|\Omega|^2}\Omega^T\mathbf{I}\Omega \; \leq \; 0,$$

which can be true only if

$$m \; - \; \frac{15}{2|\lambda|^4}\lambda^T\mathbf{I}\lambda \; \leq \; 0,$$

or

$$\frac{15}{2m}\bar{\lambda}^T\bar{\mathbf{I}}\bar{\lambda} \; \geq \; 1,
\tag{6.4.17}$$

where

$$\bar{\lambda} = \frac{\lambda}{|\lambda|}, \qquad \bar{\mathbf{I}} = \int \frac{|\bar{\mathbf{q}}|^2}{|\lambda|^2}dm(\bar{\mathbf{q}})\,\mathbf{1} \; - \; \int \frac{\bar{\mathbf{q}}\bar{\mathbf{q}}^T}{|\lambda|^2}dm(\bar{\mathbf{q}}).$$

It is easy to see that for large λ, (6.4.17) is not satisfied. In particular, to satisfy (6.4.17), the ratio $\frac{|\bar{\mathbf{q}}|^2}{|\lambda|^2}$ must be greater than $\frac{1}{15}$. But for typical artificial satellites, this ratio is approximately 10^{-10}. For the motion of moon around the earth, it is approximately 1.6×10^{-5}. Thus we have shown that for the practical case of large orbit radii, the

24 relative equilibria (for the *second-order approximate model*) are the only relative equilibria. This conclusion is of special interest since we have constructed a numerical example (the "molecule" in Figure 6.3.2) in which the *exact* model has *no* great circle relative equilibria.

6.5. Stability of Relative Equilibria in the Approximate Models

In this section, we study the stability properties of the relative equilibria for the approximate models discussed in Section 6.4. For both order zero and order two cases, the triple (Π, λ, μ) is a relative equilibrium if the three vectors are along the three principal axes. Without loss of generality we let

$$\Pi_e = |\Pi| \, e_1 = I_1 |\Omega| e_1,$$

$$\lambda_e = |\lambda| \, e_2, \qquad (6.5.1)$$

$$\mu_e = \frac{m|\Pi||\lambda|}{I_1} \, e_3 = m|\Omega||\lambda| \, e_3,$$

where $|\Omega|$ and $|\lambda|$ are related through the appropriate form of the Kepler frequency formula, and

$$\mathbf{I} \, e_1 = I_1 e_1,$$

$$\mathbf{I} \, e_2 = I_2 e_2,$$

$$\mathbf{I} \, e_3 = I_3 e_3.$$

We shall examine the stability of this relative equilibrium in various cases determined by the relative magnitudes of the principal moments of inertia I_i.

For the *order zero* reduced dynamics (6.4.4), the energy-Casimir method of Chapter 5 is inconclusive since the second variation of the energy-Casimir function is only positive semi-definite (has a zero eigenvalue). We linearize the system around the relative equilibrium (6.5.1). Let

$$\delta x = (\delta \Pi_1, \ \delta \Pi_2, \ \delta \Pi_3, \ \delta \lambda_1, \ \delta \lambda_2, \ \delta \lambda_3, \ \delta \mu_1, \ \delta \mu_2, \ \delta \mu_3)^T. \qquad (6.5.2)$$

We have the linearized system

$$\dot{\delta x} = A \, \delta x,$$

where A is the matrix,

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{|\lambda|}{I_3} & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{|\Pi|}{I_1} & 0 & \frac{1}{m} & 0 \\
0 & 0 & 0 & 0 & -\frac{|\Pi|}{I_1} & 0 & 0 & 0 & \frac{1}{m} \\
0 & -\frac{m|\Pi||\lambda|}{I_1 I_2} & 0 & -\frac{mGM}{|\lambda|^3} & 0 & 0 & 0 & 0 & 0 \\
\frac{m|\Pi||\lambda|}{I_1^2} & 0 & 0 & 0 & \frac{2mGM}{|\lambda|^3} & 0 & 0 & 0 & \frac{|\Pi|}{I_1} \\
0 & 0 & 0 & 0 & 0 & -\frac{mGM}{|\lambda|^3} & 0 & -\frac{|\Pi|}{I_1} & 0
\end{pmatrix}.
$$

with

$$
A_{23} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)|\Pi|, \quad A_{32} = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)|\Pi|.
$$

By the frequency formula (6.4.7), we can write A in the form of

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{I_3 - I_1}{I_3}|\Omega| & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{I_2 - I_1}{I_2}|\Omega| & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{|\lambda|}{I_3} & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & |\Omega| & 0 & \frac{1}{m} & 0 \\
0 & 0 & 0 & 0 & -|\Omega| & 0 & 0 & 0 & \frac{1}{m} \\
0 & -\frac{m|\Omega||\lambda|}{I_2} & 0 & -m|\Omega|^2 & 0 & 0 & 0 & 0 & 0 \\
\frac{m|\Omega||\lambda|}{I_1} & 0 & 0 & 0 & 2m|\Omega|^2 & 0 & 0 & 0 & |\Omega| \\
0 & 0 & 0 & 0 & 0 & -m|\Omega|^2 & 0 & -|\Omega| & 0
\end{pmatrix}.
$$

Denote the upper left 3×3 matrix by A_1 and the lower right 6×6 matrix by A_2. It can be shown that

$$
p_1(s) \stackrel{\triangle}{=} det(sI - A_1) = s\left(s^2 + \frac{I_3 - I_1}{I_3}\frac{I_2 - I_1}{I_2}|\Omega|^2\right),
$$

$$
p_2(s) \stackrel{\triangle}{=} det(sI - A_2) = s^2\left(s^2 + |\Omega|^2\right)^2.
$$

The characteristic polynomial of A is $p(s) = p_1(s)p_2(s)$. It can be further verified that the minimal polynomial of A is

$$m(s) = s^2 \left(s^2 + \frac{I_3 - I_1}{I_3} \frac{I_2 - I_1}{I_2} |\Omega|^2 \right) \left(s^2 + |\Omega|^2 \right).$$

The occurrence of a repeated root of the minimal polynomial at $s = 0$ implies linear instability of the relative equilibrium (6.5.1) for the *order zero* approximate model (See Gantmacher [22], Theorem 3, pp. 129). Alternatively, the one parameter family of "uniformly spinning solutions" given by (6.4.9) represents a perturbation of the relative equilibrium (6.5.1) that departs any small neighborhood of the relative equilibrium in finite time, and hence we have instability. We note that this conclusion is independent of the relative magnitudes of the I_i's.

REMARK 6.5.1

The projection of (Π, λ, μ) to the space of Π projects the *order zero* dynamics to the usual rigid body dynamics. For this *projected dynamics*, the equilibria in which the vector Π is along the maximum or minimum principal axes are stable.

∎

We now study the stability of relative equilibria of the *order two* reduced dynamics (6.4.5). For the relative equilibrium (6.5.1), we have the following identity, cf. (6.4.12),

$$m|\Omega|^2 = \frac{GM}{|\lambda|^3} \left(m + \frac{3}{2|\lambda|^2} tr(\mathbf{I}) - \frac{9}{2|\lambda|^2} I_2 \right). \tag{6.5.3}$$

Now we discuss sufficient conditions for the stability of these 24 relative equilibria. We apply the energy-Casimir method discussed in Section 5.2. The general form of the energy-Casimir functional for our case is

$$\tilde{H}_\phi = \frac{1}{2}\Pi^T \mathbf{I}^{-1}\Pi + \frac{1}{2}\frac{|\mu|^2}{m} - \frac{mGM}{|\lambda|} - \frac{GM}{2|\lambda|^3}tr(\mathbf{I}) + \frac{3GM}{2|\lambda|^5}\lambda^T \mathbf{I}\lambda + \phi\left(\frac{1}{2}|\Pi + \lambda \times \mu|^2 \right). \tag{6.5.4}$$

The first variation of $\tilde{H}_\phi$ can be found as $\delta\tilde{H}_\phi(\Pi, \lambda, \mu) = \nabla\tilde{H}_\phi \cdot \delta x$, where

$$\nabla\tilde{H}_\phi(\Pi, \lambda, \mu) = \begin{pmatrix} \mathbf{I}^{-1}\Pi + \phi'\mathbf{n} \\ \frac{GM}{|\lambda|^3}\left(m + \frac{3tr\mathbf{I}}{2|\lambda|^2} - \frac{15\lambda^T \mathbf{I}\lambda}{2|\lambda|^4} \right)\lambda + \frac{3GM}{|\lambda|^5}\mathbf{I}\lambda + \phi'\hat{\mu}\mathbf{n} \\ \frac{\mu}{m} - \phi'\hat{\lambda}\mathbf{n} \end{pmatrix},$$

140

and δx is as in (6.5.2), and,

$$\mathbf{n} \;=\; \Pi + \lambda \times \mu.$$

The matrix representation of the second variation of $\tilde{H}_\phi$ is

$$\nabla^2 \tilde{H}_\phi(\Pi, \lambda, \mu) \;\triangleq$$

$$\begin{pmatrix} \mathbf{I}^{-1} + \phi'\mathbf{1} + \phi''\mathbf{nn}^T & -\phi'\hat{\mu} - \phi''\mathbf{nn}^T\hat{\mu} & \phi'\hat{\lambda} + \phi''\mathbf{nn}^T\hat{\lambda} \\[2mm] \phi'\hat{\mu} + \phi''\hat{\mu}\mathbf{nn}^T & \dfrac{GM}{|\lambda|^3}\left(m + \dfrac{3tr\mathbf{I}}{2|\lambda|^2} - \dfrac{15\lambda^T\mathbf{I}\lambda}{2|\lambda|^4}\right)\mathbf{1} & -\phi'\hat{\mathbf{n}} + \phi'\hat{\mu}\hat{\lambda} \\ & -\dfrac{3GM}{|\lambda|^5}\left(m + \dfrac{5tr\mathbf{I}}{2|\lambda|^2} - \dfrac{35\lambda^T\mathbf{I}\lambda}{2|\lambda|^4}\right)\lambda\lambda^T & +\phi''\hat{\mu}\mathbf{nn}^T\hat{\lambda} \\ & -\dfrac{15GM}{|\lambda|^7}\lambda\lambda^T\mathbf{I} - \dfrac{15GM}{|\lambda|^7}\mathbf{I}\lambda\lambda^T & \\ & +\dfrac{3GM}{|\lambda|^5}\mathbf{I} - \phi'\hat{\mu}\hat{\mu} - \phi''\hat{\mu}\mathbf{nn}^T\hat{\mu} & \\[2mm] -\phi'\hat{\lambda} - \phi''\hat{\lambda}\mathbf{nn}^T & \phi'\hat{\mathbf{n}} + \phi'\hat{\lambda}\hat{\mu} + \phi''\hat{\lambda}\mathbf{nn}^T\hat{\mu} & \dfrac{1}{m}\mathbf{1} - \phi'\hat{\lambda}\hat{\lambda} \\ & & -\phi''\hat{\lambda}\mathbf{nn}^T\hat{\lambda} \end{pmatrix}.$$

In the above formulae, ϕ' represents its value at $|\mathbf{n}|^2/2$, and the same convention is applied to ϕ''. Now we find the variations at $(\Pi_e, \lambda_e, \mu_e)$. By using (6.5.3), we have

$$\nabla \tilde{H}_\phi(\Pi_e, \lambda_e, \mu_e) \;=\; (1 + \phi'K)\begin{pmatrix} |\Omega|e_1 \\ m|\lambda||\Omega|^2 e_2 \\ |\lambda||\Omega|e_3 \end{pmatrix},$$

where

$$K \;=\; I_1 + m|\lambda|^2.$$

Thus in order for the first variation to vanish, we require $1 + \phi'K = 0$, or

$$\phi' \;=\; -\frac{1}{K}. \tag{6.5.5}$$

Substituting these values in the second variation formula, we get a 9×9 symmetric matrix $F = \nabla^2 \tilde{H}_\phi(\Pi_e, \lambda_e, \mu_e)$ with nonzero components,

$$F_{11} = \frac{R - I_1}{I_1 R}, \quad F_{15} = -\frac{m|\Omega||\lambda|}{R}, \quad F_{19} = -\frac{|\lambda|}{R},$$

$$F_{22} = \frac{K - I_2}{I_2 K}, \quad F_{24} = \frac{m|\Omega||\lambda|}{K},$$

$$F_{33} = \frac{K - I_3}{I_3 K}, \quad F_{37} = \frac{|\lambda|}{K},$$

$$F_{44} = m|\Omega|^2 \frac{I_1}{K} + \frac{3GM}{|\lambda|^5}(I_1 - I_2), \tag{6.5.6}$$

$$F_{55} = -m|\Omega|^2(4 + \frac{m|\lambda|^2}{R}) + \frac{2mGM}{|\lambda|^3}, \quad F_{59} = -|\Omega|(1 + \frac{m|\lambda|^2}{R}),$$

$$F_{66} = m|\Omega|^2 + \frac{3GM}{|\lambda|^5}(I_3 - I_2), \quad F_{68} = |\Omega|,$$

$$F_{77} = \frac{I_1}{mK}, \quad F_{88} = \frac{1}{m}, \quad F_{99} = \frac{R - m|\lambda|^2}{mR},$$

where

$$\frac{1}{R} = \frac{1}{K} - \phi'' K^2 |\Omega|^2, \tag{6.5.7}$$

With the lower triangular matrix L given by nonzero elements as,

$$L_{42} = -\frac{I_2 m|\Omega||\lambda|}{K - I_2}, \tag{6.5.8a}$$

$$L_{51} = \frac{m|\Omega||\lambda|I_1}{R - I_1}, \tag{6.5.8b}$$

$$L_{73} = -\frac{I_3|\lambda|}{K - I_3}, \tag{6.5.8c}$$

$$L_{86} = -\frac{|\Omega|}{F_{66}}, \tag{6.5.8d}$$

$$L_{91} = \frac{I_1|\lambda|}{R - I_1}\left(m|\Omega|L_{95} + 1\right), \tag{6.5.8e}$$

$$L_{95} = \frac{U_{59}}{D_{55}}, \tag{6.5.8f}$$

$$U_{59} = -|\Omega|\left(1 + \frac{K - I_1}{R - I_1}\right), \tag{6.5.8g}$$

$$D_{55} = m|\Omega|^2\left(4 + \frac{K - I_1}{R - I_1}\right) - \frac{2mGM}{|\lambda|^3}, \tag{6.5.8h}$$

and all the diagonal elements are 1. we can transform F into a diagonal matrix

$$D = L F L^T,$$

where

$$D = \text{diag}\left\{ \frac{R - I_1}{I_1 R}, \ \frac{K - I_2}{I_2 K}, \ \frac{K - I_3}{I_3 K}, \ (I_1 - I_2)D_{44}, \ -D_{55}, F_{66}, \right.$$

$$\left. \frac{I_1 - I_3}{m(K - I_3)}, \ (I_3 - I_2)D_{88}, \ \frac{U_{59}^2}{D_{55}} + \frac{R - K}{m(R - I_1)} \right\},$$

with

$$D_{44} = \frac{m|\Omega|^2}{K - I_2} + \frac{3GM}{|\lambda|^5}$$

$$D_{88} = \frac{3GM}{m|\lambda|^5 F_{66}}.$$

(Since congruence transformations preserve the *matrix inertia*, we can read off the number of negative eigenvalues of F from D.)

We shall now consider the case in which,

$$I_1 > I_3 > I_2.$$

To have stability from the energy-Casimir method, we require that all the eigenvalues of F (equivalently of D) be positive. This holds if

$$\frac{R - I_1}{R} > 0, \tag{6.5.9a}$$

$$D_{55} < 0, \tag{6.5.9b}$$

and

$$\frac{U_{59}^2}{D_{55}} + \frac{R - K}{m(R - I_1)} > 0. \tag{6.5.9c}$$

For (6.5.7), we have

$$\frac{R - I_1}{R} = 1 - \frac{I_1}{R} = \frac{m|\lambda|^2}{K} + \phi'' I_1 K^2 |\Omega|^2.$$

Thus (6.5.9a) holds if $\phi'' > 0$. We next consider (6.5.9b). By the definition (6.5.8h) of D_{55} and the frequency formula (6.5.3), we get

$$D_{55} = m|\Omega|^2 \left(4 + \frac{K - I_1}{R - I_1} - \frac{2}{1 + \frac{3}{2m|\lambda|^2} tr\mathbf{I} - \frac{9}{2m|\lambda|^2} I_2} \right).$$

Let

$$\epsilon \triangleq \frac{3}{2m|\lambda|^2}(I_1 + I_3 - 2I_2).$$

For the case under consideration $\epsilon > 0$ and for $|\lambda|$ large compared to the typical dimensions of the body, ϵ is small. The other term,

$$\frac{K - I_1}{R - I_1} = \frac{m|\lambda|^2\left(1/K - \phi''K^2|\Omega|^2\right)}{m|\lambda|^2/K + \phi''I_1K^2|\Omega|^2}.$$

For $|\lambda|$ large and ϕ'' large enough,

$$\frac{K - I_1}{R - I_1} \simeq \frac{m|\lambda|^2\left(-\phi''K^2|\Omega|^2\right)}{\phi''I_1K^2|\Omega|^2} = -\frac{m|\lambda|^2}{I_1} \triangleq -\theta. \tag{6.5.10}$$

Thus

$$D_{55} \simeq m|\Omega|^2\left(4 - \frac{2}{1+\epsilon} - \theta\right),$$

for $|\lambda|$ large and ϕ'' large enough. Since $\theta >> 4$, we have $D_{55} < 0$. This is (6.5.9b). Now we look at (6.5.9c). It is easy to see that *if we show*

$$U_{59}^2 + \frac{R - K}{m(R - I_1)}D_{55} < 0, \tag{6.5.11}$$

then together with (6.5.9b), we have (6.5.9c). From (6.5.10), for $|\lambda|$ large and ϕ'' large enough,

$$U_{59} \simeq -|\Omega|(1 - \theta).$$

From the definition (6.5.7) of R, we have

$$\frac{R - K}{R} = \phi''K^3|\Omega|^2.$$

Thus

$$\frac{R - K}{m(R - I_1)} = \frac{\phi''K^3|\Omega|^2}{\frac{m|\lambda|^2}{K} + \phi''I_1K^2|\Omega|^2} \simeq \frac{K}{I_1} = 1 + \theta,$$

for $|\lambda|$ large and ϕ'' large enough. Now we verify (6.5.11). Under the same condition,

$$\begin{aligned}
U_{59}^2 + \frac{R - K}{m(R - I_1)}D_{55} &\simeq |\Omega|^2(1 - \theta)^2 + \frac{1}{m}(1 + \theta)m|\Omega|^2\left(4 - \frac{2}{1+\epsilon} - \theta\right) \\
&= |\Omega|^2\left(5 - \frac{2}{1+\epsilon} + (1 - \frac{2}{1+\epsilon})\theta\right) \\
&\simeq -|\Omega|^2\theta \\
&< 0.
\end{aligned}$$

Thus (6.5.7) hold for $|\lambda|$ large and ϕ'' large enough. We have the following theorem.

STABILITY THEOREM 6.5.2

For the *order two* approximate model, the relative equilibrium

$$\mathbf{I}\,\Pi \;=\; I_1\Pi$$

$$\mathbf{I}\,\lambda \;=\; I_2\lambda$$

$$\mathbf{I}\,\mu \;=\; I_3\mu$$

is stable if $|\lambda|$ is sufficiently large and,

$$I_1 \;>\; I_3 \;>\; I_2.$$

∎

This shows that the relative equilibrium in which the body center of mass traverses a circular orbit, the angular velocity lies along the principal axis of the body with the largest associated moment of inertia (minor axis of the ellipsoid of inertia), and the radius vector is aligned to the principal axis with the least associated moment of inertia (major axis of the ellipsoid of inertia), is a stable relative equilibrium.

HISTORICAL REMARK 6.5.3

A similar theorem appears in Beletskii's book [8], pp. 94–102. Beletskii uses a spatial/inertial model of the coupling between translational and rotational motion and presents arguments based on a Lyapunov–Chetayev approach [13], and uses in effect the variational equations about the stationary motion. In contrast, here we make consistent use of modern hamiltonian methods and reduced variables. The methods of this paper yield a nonlinear stability theorem and generalize to nonrigid and other complex configurations. See for instance the examples considered in [38], [44], [33], [41], [42], [56].

∎

The above discussion demonstrated that it is sometimes not straightforward to explicitly find an appropriate function ϕ in the energy-Casimir method. In Section 5.3,

we describe a more classical characterization of relative equilibria as critical points of the constrained variational principle,

$$\min \quad \tilde{H}_2(\Pi, \lambda, \mu) \tag{6.5.12}$$

$$\text{subject to} \quad C(\Pi, \lambda, \mu) = constant$$

where $\tilde{H}_2$ is the Hamiltonian (6.4.3) and C is the Casimir $\frac{1}{2}|\Pi + \lambda \times \mu|^2$. The associated first-order conditions coincide with the characterization (LM) of relative equilibria, with the unknown constant α being interpreted as a Lagrange multiplier.

The Lagrangian (in the sense of optimization theory) associated with the above constrained variational principle is recovered if in (6.5.4) we take $\phi(x) = -\alpha x$. Consequently the second variation can be recovered as a special case of that calculated in (6.5.6). When ϕ is linear, $\phi'' = 0$, and consequently, $R = K$. Therefore for the second variation F^α of $H - \alpha C$, cf. (6.5.6), we have the matrix elements

$$F_{11}^\alpha = \frac{K - I_1}{I_1 K}, \quad F_{15}^\alpha = -\frac{m|\Omega||\lambda|}{K}, \quad F_{19}^\alpha = -\frac{|\lambda|}{K},$$

$$F_{22}^\alpha = \frac{K - I_2}{I_2 K}, \quad F_{24}^\alpha = \frac{m|\Omega||\lambda|}{K},$$

$$F_{33}^\alpha = \frac{K - I_3}{I_3 K}, \quad F_{37}^\alpha = \frac{|\lambda|}{K},$$

$$F_{44}^\alpha = m|\Omega|^2 \frac{I_1}{K} + \frac{3GM}{|\lambda|^5}(I_1 - I_2), \tag{6.5.13}$$

$$F_{55}^\alpha = -m|\Omega|^2(4 + \frac{m|\lambda|^2}{K}) + \frac{2mGM}{|\lambda|^3}, \quad F_{59}^\alpha = -|\Omega|(1 + \frac{m|\lambda|^2}{K}),$$

$$F_{66}^\alpha = m|\Omega|^2 + \frac{3GM}{|\lambda|^5}(I_3 - I_2), \quad F_{68}^\alpha = |\Omega|,$$

$$F_{77}^\alpha = \frac{I_1}{mK}, \quad F_{88}^\alpha = \frac{1}{m}, \quad F_{99}^\alpha = \frac{K - m|\lambda|^2}{mK},$$

A comparatively simple Gaussian elimination then reveals that F^α is congruent to

$$D^\alpha = diag\left\{ \frac{K - I_1}{I_1 K}, \frac{K - I_2}{I_2 K}, \frac{K - I_3}{I_3 K}, (I_1 - I_2)D_{44}, -D_{55}^\alpha, F_{66}, \right.$$

$$\left. \frac{I_1 - I_3}{m(K - I_3)}, (I_3 - I_2)D_{88}, \frac{4|\Omega|^2}{D_{55}^\alpha} \right\},$$

where

$$D_{55}^\alpha = 5m|\Omega|^2 - \frac{2mGM}{|\lambda|^3}$$

146

and the other coefficients are as defined previously. From the identity (6.5.3), we have further that

$$D_{55}^{\alpha} = m|\Omega|^2 \left(5 - \frac{2}{1+\epsilon} \right).$$

For $|\lambda|$ sufficiently large, the expressions for the various coefficients reveal that K is large and positive, and D_{44}, D_{55}^{α}, F_{66} and D_{88} are all positive. Consequently the signs of the entries of D^{α} are determined by the signs of the entries

$$\{ + , + , + , (I_1 - I_2) , - , + , (I_1 - I_3) , (I_3 - I_2) , + \}.$$

We shall restrict attention to satellites in which the inertias are distinct so that F^{α} is nonsingular. Otherwise additional symmetries arise, and the analysis is slightly more complicated. There are six cases of distinct inertias. See Table 6.5.2, in which the number of negative eigenvalues of F^{α} is shown in each case, and each case is assigned a reference number in parentheses.

Table 6.5.2. Rigid Body Inertia Combinations

	I_2 min	I_2 middle	I_2 max
$I_1 > I_3$	(1) 1	(2) 2	(3) 3
$I_1 < I_3$	(4) 2	(5) 3	(6) 4

According to Theorem 5.3.2, it suffices to analyze whether the condition

$$\langle h, F^{\alpha} h \rangle > 0, \quad \forall h \neq 0 \text{ and } \langle \nabla C(\Pi_e, \lambda_e, \mu_e), h \rangle = 0 \tag{6.5.14}$$

is satisfied. Because the subspace of admissible variations h has codimension 1, condition (6.5.14) cannot hold whenever F^{α} has two or more negative eigenvalues. Accordingly the only case in which (6.5.14) might hold is case (1), $I_1 > I_3 > I_2$, in which F^{α} is nonsingular and has precisely one negative eigenvalue.

To analyze (6.5.14) in case (1), we shall apply a general result bearing upon families of extremals to variational principles. Notice that (6.5.1) actually defines a one-parameter family of relative equilibria which can be regarded as being parametrized

by the magnitude of the radius of the orbit, i.e. $|\lambda|$. But the multiplier α is related to $|\lambda|$ through $\alpha = \frac{1}{I_1 + m|\lambda|^2}$ (cf. (6.5.5)), so the family can also be parametrized by the multiplier α. Along this family the Casimir can be written, using (6.5.1) and (6.5.3), as

$$
\begin{aligned}
\frac{1}{2}|\Pi + \lambda \times \mu|^2 &= \frac{1}{2}|\Omega|^2(I_1 + m|\lambda|^2)^2 \\
&= \frac{GM(I_1 + m|\lambda|^2)^2}{m|\lambda|^5}\left(m|\lambda|^2 + \frac{3}{2}(I_1 - 2I_2 + I_3)\right). \quad (6.5.15)
\end{aligned}
$$

Consequently, for $|\lambda|$ large the Casimir is an increasing function of $|\lambda|$ along the family of relative equilibria, and consequently a *decreasing* function of α. We may now apply the aforementioned result.

<u>LEMMA 6.5.4</u> (Maddocks [44], Lemma 5.2, pp. 316)

Suppose a family of variational principles of the type (6.5.12) have a family of critical points $x_e(\alpha)$ parametrized by the multiplier α. Moreover, suppose that the second variation at a particular extremal is nonsingular with one negative eigenvalue. Then the second-order sufficient conditions (6.5.14) at that extremal are satisfied if and only if the constraint C is a *decreasing* function of the multiplier α at that parameter value.

<u>COROLLARY 6.5.5</u>

Solutions (6.5.1) in case (1), $I_1 > I_3 > I_2$ are Lyapunov stable for all $|\lambda|$ sufficiently large.

Proof

It has been shown that the hypotheses of the previous Lemma hold, and that $C(\alpha)$ is decreasing for relative equilibria with $|\lambda|$ sufficiently large. Thus condition (6.5.14) holds and Theorem 5.3.2 then applies to provide the desired result.

∎

The configuration of this stable relative equilibrium could be depicted in Figure 6.5.1.

Accordingly we have rederived the Stability Theorem proved in Theorem 6.5.2. We notice that here we have not proven instability in the *order two* model. Actually
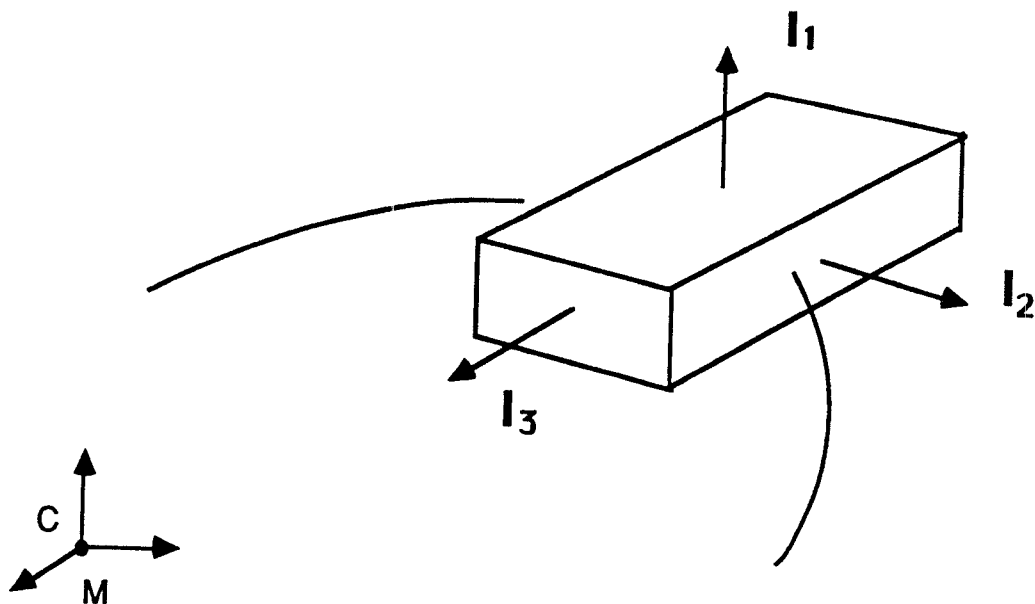
Figure 6.5.1. Configuration of the Stable Relative Equilibrium

the results of Maddocks [45](Section 5) can be applied to show that for large $|\lambda|$ the relative equilibria in any of the cases (2),(4),(6) in Table 1, are dynamically unstable. An outline of the analysis is that when F^α has an even number of negative eigenvalues and $C(\alpha)$ is a decreasing function, then the linearized dynamics must possess an unstable real eigenvalue.

We remark that the energy-momentum method developed in Section 5.4 can be also applied in the order two approximate model. The same sufficient conditions for stability could be obtained. Since we are more interested in applying that method for the problem including momentum wheels on the rigid body, we present it in more detail in the following chapter.

149

CHAPTER VII

Gyroscopic Control

In Chapter 3, we introduced the notion of a gyroscopic system with symmetry. One example was discussed in Chapter 4, namely, the multibody analog of the dual-spin problem. Equipped with the tools for stability analysis developed in Chapter 5, we investigate several key examples that incorporate gyroscopic effects. We give precise sufficient conditions for stability in each example. It is useful to view each of these examples as resulting from gyroscopic feedback as defined below.

7.1. Gyroscopic Feedback

The notion of *gyroscopic control* is isolated here to highlight the role of the gyroscopic term from the viewpoint of designing control algorithms. A simple mechanical system with symmetry with exterior forces can be transformed into a gyroscopic system with symmetry by using suitable feedback laws, that we refer to as *gyroscopic feedback*. This process is described in the following theorem.

THEOREM 7.1.1

Consider a simple mechanical system with symmetry, (Q, K, V, G) in which the riemannian metric is given by, in local coordinates,

$$K(x)(v, \ w) \ = \ v^T M(x) w. \tag{7.1.1}$$

The exterior force exerted on the system is denoted by a horizontal 1-form $(\alpha, \ 0)$. Let $\tilde{Y}$ be *any* G-invariant 1-form on Q. Then, with the feedback law,

$$\alpha(x, \ v) \ = \ -\left(\frac{\partial \tilde{Y}}{\partial x}(x) - \frac{\partial \tilde{Y}}{\partial x}^T (x) \right) \ v, \tag{7.1.2}$$

the closed-loop system becomes a gyroscopic system with symmetry (Q, K, Y, V, G) where,

$$Y(x) = M(x)^{-1} \bar{Y}(x). \tag{7.1.3}$$

Proof

The dynamical equations for (Q, K, V, G) with exterior force can be found to be,

$$M(x) \cdot \ddot{x} = \mathcal{T} \cdot \dot{x} \cdot \dot{x} - \frac{\partial V}{\partial x}(x) + \alpha. \tag{7.1.4}$$

where $\mathcal{T}$ is defined in (3.2.6b). With the feedback law (7.1.2), it is then easy to see that (7.1.4) becomes (3.2.7) which, in turn, corresponds to a system with Lagrangian in the form of (3.2.2). With the transformation rule (7.1.3), the system can be further identified as a gyroscopic system with symmetry, (Q, K, Y, V, G).

∎

Accordingly, we have a family of gyroscopic feedback laws induced by G-invariant 1-form. The techniques used in analyzing gyroscopic systems can then be applied. In particular, the methods for stability analysis based on the energy-Casimir method, the Lagrange-multiplier method, and the energy-momentum method as discussed in Chapter 5 are applicable. The gyroscopic term affects the dynamical behavior in many aspects, as will be seen in the following sections. As a consequence, a suitable gyroscopic feedback may be chosen to fulfill design objectives. Much work remains to be done on general methods for choosing $\ddot{Y}$.

7.2. Rigid Body Attitude Control

We first consider the well-treated single rigid body dual-spin problem, see e.g. [36] [3]. The system consists of a rigid body, the platform, with three symmetric rotors, the driven rotors, spinning at constant speeds relative to the platform along the principal axes of the platform. See Figure 7.2.1. Following similar notations in Chapter 4, let $B \in SO(3)$ denote the attitude of the platform with Ω its instantaneous body angular velocity. Let I, I_{S_i}, $i = 1, 2, 3$ be the moments of inertia of the platform and the driven rotors, respectively. The constant relative angular velocities of the driven rotors are denoted by a vector $\dot{\Theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$. The Lagrangian of the system takes the form, cf. (4.3.6),

$$L(B, \ \Omega) \ = \ \frac{1}{2} < \Omega, \ \mathrm{J}\Omega >_E \ + \ < \Omega, \ \mathrm{I}^S \dot{\Theta} >_E, \qquad (7.2.1)$$

where

$$\mathrm{J} \ = \ \mathrm{I} \ + \ \sum_{i=1}^{3} \mathrm{I}_{S_i}, \qquad (7.2.2a)$$

$$\mathrm{I}^S \ = \ \mathrm{diag}\big((\mathrm{I}_{S_1})_1 \ , (\mathrm{I}_{S_2})_2 \ , (\mathrm{I}_{S_3})_3\big). \qquad (7.2.2b)$$

Namely, J is the total moment of inertia of the platform with locked driven rotors and I^S is a diagonal matrix consisting of the components of the moments of inertia of the driven rotors along their spinning axes. Both matrices are expressed in the body frame.
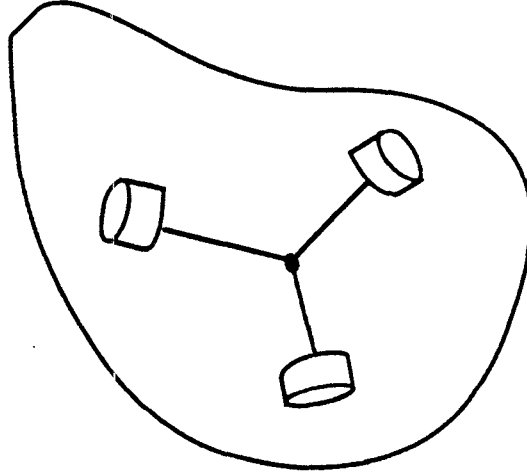

Figure 7.2.1. Dual-spin Problem

Letting $l = \mathrm{I}^S \dot{\Theta}$, (7.2.1) can be further written as

$$L(B, \ \Omega) \ = \ \frac{1}{2} < \Omega, \ \mathrm{J}\Omega > \ + \ < \Omega, \ l >, \qquad (7.2.3)$$

by dropping the subscript E for the Euclidean inner product. This system belongs to the category of gyroscopic systems with symmetry with the following entities,

152

$$Q = SO(3),$$

$$K(B\hat{u}_1, B\hat{u}_2) = <u_1, Ju_2>,$$

$$Y(B) = B\widehat{J^{-1}l}, \tag{7.2.4}$$

$$V(B) = 0,$$

$$G = SO(3),$$

where $B\hat{u}_1, B\hat{u}_2 \in T_B SO(3)$. The group action here is, cf. (3.5.6),

$$G \times Q \to Q,$$

$$(R, B) \mapsto RB,$$

with the infinitesimal generator, cf. (3.5.7),

$$\xi_Q(B) = \hat{\xi} B,$$

for $\xi \in \mathcal{G}$. It is straightforward to check that K, Y are both invariant under the group action. To find a relative equilibrium, we apply the principle of symmetric criticality, Algorithm 3.4.4. Fix $\xi \in \mathcal{G}$. The augmented potential V_ξ is found to be

$$
\begin{aligned}
V_\xi(B) &= - \ll Y(B), \xi_Q(B) \gg - \frac{1}{2} \ll \xi_Q(B), \xi_Q(B) \gg, \\
&= - <J^{-1}l, JB^T\xi> - \frac{1}{2} <B^T\xi, JB^T\xi>, \tag{7.2.5} \\
&= - <\xi, Bl> - \frac{1}{2} <\xi, BJB^T\xi>.
\end{aligned}
$$

A critical point B_e of this function gives us the relative equilibrium $(B_e, \hat{\xi} B_e) \in TQ$. Letting $\nu = B^T\xi$, the problem of finding the critical points of V_ξ is equivalent to finding the critical points of

$$\bar{V}_\xi(\nu) = - <\nu, l> - \frac{1}{2} <\nu, J\nu>, \tag{7.2.6}$$

subject to the constraint

$$|\nu| = |\xi| = \text{constant}.$$

This function is in fact the one defined in (3.4.8) as a function on Q/G_ξ. Since the group here is $SO(3)$, as discussed in Section 5.4, $\bar{V}_\xi$ is sufficient to determine stability. We

153

may now vary the constant vector l and see how the phase portrait is affected from $\tilde{V}_\xi$. In particular, for

$$l = (l_1, 0, 0), \quad \text{and} \quad \mathbf{J} = \text{diag}(J_1, J_2, J_3), \tag{7.2.7}$$

we plot the function $\tilde{V}_\xi$ on the sphere $|\nu| = $ constant for different l_1 in Figure 7.2.2.

Notice that in Figure 7.2.2(a), the minimum or maximum points of $\tilde{V}_\xi$ correspond to rotation about the axes with largest or smallest moment of inertia, respectively. These are stable relative equilibria. The intermediate axis corresponds to the saddles of $\tilde{V}_\xi$. These reconstruct classical stability results for rigid body dynamics. It may be observed that the number of critical points of $\tilde{V}_\xi$ reduces from six in (a) to two in (d) with increasing l_1. Also the critical points change their properties with different l_1. This is the phenomenon discussed in [34] [36]. For sufficiently large l_1, the state that the platform rotates about the axis l could be made to be the only stable critical point. Accordingly, with a suitable damping mechanism, attitude acquisition of the platform could be achieved through this process. By varying l_1, we could thus "shape" the phase portrait as we desire. This exhibits the essential idea behind *gyroscopic control*. Namely, by varying the gyroscopic field Y, the phase portrait, or the structure of the dynamical behavior, could be controlled.

Now we apply the energy-momentum method to this example to get some quantitative understanding. Since for this problem, cf. (7.2.4), $G = Q$, we need only consider the Arnold block for stability, cf. Corollary 5.4.12. Also since $G = SO(3)$, the analysis in (5.4.51) could be applied. Accordingly, we check the positive definiteness of $I^o_{lock}(B_e)^{-1} - \frac{1}{\lambda}1$ in any direction except ξ. Here the locked inertia dyadic can be found through, for $\hat{\xi}$, $\hat{\eta} \in so(3)$,

$$\langle \hat{\xi}, I_{lock}(B)\hat{\eta} \rangle = \ll \hat{\xi}B, \hat{\eta}B \gg = < \xi, BJB^T\eta > .$$

It follows that, cf. (5.4.46),

$$I^o_{lock}(B) = B \mathbf{J} B^T. \tag{7.2.8}$$

The element in $\mathcal{G}^*$ induced by gyroscopic field Y is determined from, for $\hat{\eta} \in so(3)$,
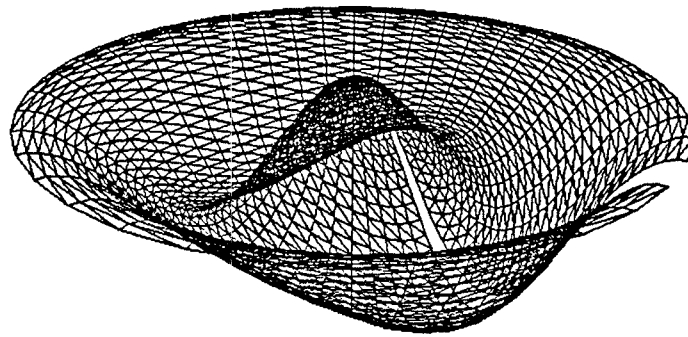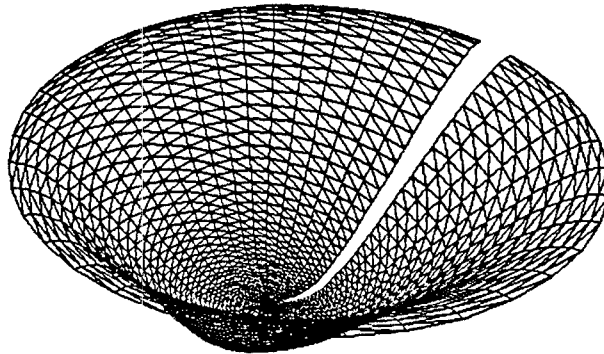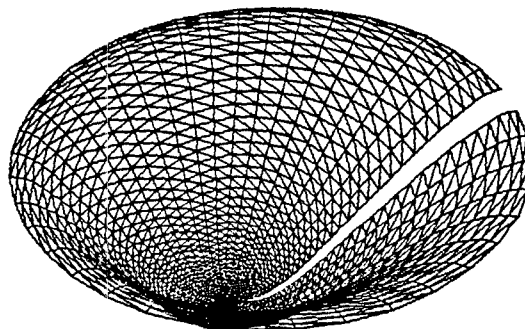
Figure 7.2.2. Plot of $\tilde{V}_\xi$

155

$$\langle \mathbf{I}_Y(B), \hat{\eta} \rangle \; = \; \ll B\widehat{J^{-1}l}, \; \hat{\eta}B \gg \; = \; < Bl, \; \eta >,$$

thus, cf. (5.4.47), the gyro-momentum is,

$$\mathbf{I}_Y^o(B) \; = \; Bl. \tag{7.2.9}$$

At relative equilibrium, cf. (5.4.49a),

$$\mu_e \; = \; B_e J B_e^T \xi \; + \; B_e l. \tag{7.2.10}$$

In order to satisfy the criticality condition, (5.4.45), we need

$$B_e J B_e^T \xi \; + \; B_e l \; = \; \lambda \, \xi. \tag{7.2.11}$$

Now let $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ denote three principal axis of the platform, and

$$\mathbf{J} \, \mathbf{e_1} \; = \; J_1 \, \mathbf{e_1}, \quad \mathbf{J} \, \mathbf{e_2} \; = \; J_2 \, \mathbf{e_2}, \quad \mathbf{J} \, \mathbf{e_3} \; = \; J_3 \, \mathbf{e_3}. \tag{7.2.12}$$

Let

$$\xi \; = \; |\xi| \, B\mathbf{e_1}, \quad l \; = \; l_1 \, \mathbf{e_1}, \quad \text{with} \quad l_1 \geq 0. \tag{7.2.13}$$
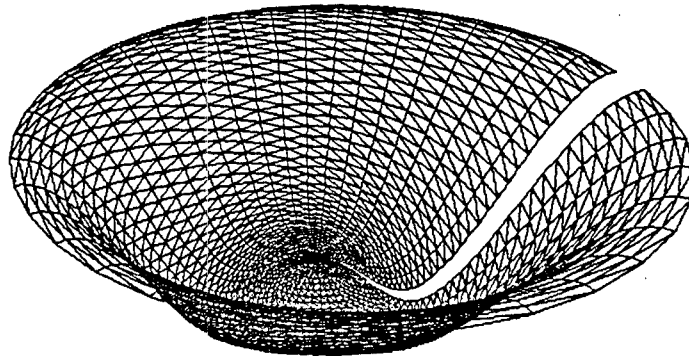
This gives rise to a relative equilibrium with

$$\lambda \; = \; \frac{J_1 |\xi| \; + \; l_1}{|\xi|}. \tag{7.2.14}$$

Since $\mathbf{I}_{lock}^o(B)^{-1} = BJ^{-1}B^T$, we have

$$\mathbf{I}_{lock}^o(B_e)^{-1} \; - \; \frac{1}{\lambda}\mathbf{1} \; = \; B_e \begin{pmatrix} \frac{1}{J_1} - \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{J_2} - \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{J_3} - \frac{1}{\lambda} \end{pmatrix} B_e^T.$$

By taking out the direction ξ, stability is ensured by the conditions

$$\frac{1}{J_2} - \frac{1}{\lambda} \; > \; 0, \quad \text{and} \quad \frac{1}{J_3} - \frac{1}{\lambda} \; > \; 0,$$

which is equivalent to, by substituting (7.2.14),

$$l_1 \; > \; (J_2 \; - \; J_1) \, |\xi|, \quad \text{and} \quad l_1 \; > \; (J_3 \; - \; J_1) \, |\xi|. \tag{7.2.15}$$

Moreover, from (7.2.10), we know that $|\mu_e| = J_1|\xi| + l_1$. Thus conditions (7.2.15) are equivalent to the conditions

$$\frac{l_1}{|\mu_e| - l_1} > \frac{J_2 - J_1}{J_1}, \qquad\qquad (7.2.16a)$$

$$\frac{l_1}{|\mu_e| - l_1} > \frac{J_3 - J_1}{J_1}. \qquad\qquad (7.2.16b)$$

These sufficient conditions for stability are useful in design/choice of l_1. For example, if we want the platform to rotate stably about its intermediate axis, we have

$$J_3 > J_1 > J_2.$$

Condition (7.2.16a) is satisfied trivially. We then design the relative angular momentum l to satisfy $l_1 > (J_3 - J_1)|\xi|$. The stable motion about its intermediate principal axis is thus obtained. In [11] [10], a variety of feedback laws were developed for rigid body dynamics with exogenous force where the closed-loop systems are of the dual-spin form in a generalized sense. While the energy-Casimir method was used there, as illustrated above, the extension of the energy-momentum method developed in this dissertation could be directly applied.

7.3. Rigid Body with Momentum Wheel in a Gravitational Field

In this section, we consider the motion of a rigid body with rotors in a central gravitational force field. This problem is a hybrid of the problems discussed in Chapter 6 and in Section 7.2. We will utilize the energy-momentum method for stability analysis and will recover the stability conditions of Chapter 6 when we specialize to the case of no driven momentum wheels.

The system configuration can be imagined by replacing the rigid body in Figure 6.1.1 by the assembly depicted in Figure 7.2.1. Following similar notations in previous discussions, the system may be put into the framework of gyroscopic systems with symmetry with the following entities, cf. (6.3.1), (7.2.4),

$$Q = SE(3),$$

$$K\big((B\hat{u}_1, v_1), (B\hat{u}_2, v_2)\big) = < u_1, Ju_2 > + m < v_1, v_2 >,$$

$$Y(B, r) = (B\widehat{J^{-1}l}, 0),$$ (7.3.1)

$$V(B, r) = -\int_B \frac{GM}{|r + B\bar{q}|} dm(\bar{q}) - \sum_{i=1}^{3} \int_{S_i} \frac{GM}{|r + B\bar{q}_i|} dm(\bar{q}_i),$$

$$G = SO(3),$$

where $\mathbf{J}$, l is defined in (7.2.2a), (7.2.3) respectively and m is the total mass of the body and rotors. The lagrangian of this system can be then written as, cf. (6.1.3), (7.2.3),

$$L(B, r, \Omega, v) = \frac{1}{2} < \Omega, \mathbf{J}\Omega > + \frac{m}{2} | v |^2 + < \Omega, l > - V(B, r).$$ (7.3.2)

Analogous to the discussions in Chapter 6, we could approximate the system by the approximations of the potential function V. In the following, we will consider the order two approximate model, namely, V is approximated by, cf. (6.4.1),

$$\bar{V}(B, r) = -\frac{mGM}{|r|} - \frac{GM}{2|r|^3} tr(\mathbf{J}) + \frac{3GM}{2|r|^5} < r, BJB^T r > .$$ (7.3.3)

The group action is the same as in (6.1.4), with the infinitesimal generator shown in (6.3.2). The locked inertia dyadic associated with the riemannian metric can be found as, cf. (6.3.3),

$$I_{lock}^o(B, r) = BJB^T - m\hat{r}\hat{r}.$$ (7.3.4)

The gyro-momentum in $\mathcal{G}^*$ induced by the gyroscopic field is the same as in Section 7.2, namely,

$$I_Y^o(B, r) = Bl.$$ (7.3.5)

The momentum mapping associated with this system is then, cf. (5.4.17), for $\eta \in \mathcal{G}$,

$$\eta \mapsto (BJB^T - m\hat{r}\hat{r}) \eta + Bl.$$ (7.3.6)

Now we apply the principle of symmetric criticality. For fixed $\xi \in \mathcal{G} = so(3) \cong \mathbb{R}^3$, the augmented potential V_ξ is

$$V_\xi(B, r) = \bar{V}(B, r) - \frac{1}{2} < \xi, (BJB^T - m\hat{r}\hat{r})\xi > - < Bl, \xi > .$$ (7.3.7)

The derivative of the augmented potential can be found as

$$DV_\xi(B, \; r) \cdot (\hat{u}B, \; v)$$

$$= \; < \frac{GM}{|r|^3}\left(m + \frac{3}{2|r|^2}\, tr(\mathbf{J}) - \frac{15}{2|r|^4}\, r^T BJB^T r\right) \; r \; - \; m\hat{\xi}\hat{r}\xi$$

$$+ \; \frac{3GM}{|r|^5}\, BJB^T r, \quad v >$$

$$+ \; < -\frac{3GM}{|r|^5}\cdot \hat{r}BJB^T r \; + \; \hat{\xi}BJB^T\xi \; + \; \hat{\xi}Bl, \quad u > . \tag{7.3.8}$$

Accordingly we get the conditions for relative equilibrium, cf. (6.3.5), (6.4.5),

$$\xi \times (BJB^T\xi \; + \; Bl) \; = \; \frac{3GM}{|r|^5}\, r \times BJB^T r,$$

$$m\xi \times (r \times \xi) \; = \; \frac{GM}{|r|^3}\left(m + \frac{3}{2|r|^2}\, tr(\mathbf{J}) - \frac{15}{2|r|^2}\, r^T BJB^T r\right) \; r \tag{7.3.9}$$

$$- \; \frac{3GM}{|r|^5}\, BJB^T r.$$

Now for $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ being three principal axes of the body with rotors, i.e. $\mathbf{J}\,\mathbf{e_i} = J_i\,\mathbf{e_i}$, for $i = 1, 2, 3$, we let, cf. (7.2.13),

$$\xi \; = \; |\xi|\, B\mathbf{e_1}, \qquad l \; = \; l_1\,\mathbf{e_1}, \quad \text{with} \quad l_1 \geq 0,$$

$$r \; = \; |r|\, B\mathbf{e_2}. \tag{7.3.10}$$

This satisfies (7.3.9) with the modified Kepler's frequency formula, cf. (6.5.3),

$$m|\xi|^2 \; = \; \frac{GM}{|r|^3}\left(m + \frac{3}{2|r|^2}\, tr(\mathbf{J}) - \frac{9}{2|r|^2}\, J_2\right). \tag{7.3.11}$$

Conditions (7.3.10) gives rise to the relative equilibria defined in (6.5.1) discussed in Section 6.5 for the case $l = 0$. With the relative equilibrium $(B_e, \; r_e)$ satisfying (7.3.10), the momentum mapping μ_e is, from (7.3.6),

$$\mu_e \; = \; (B_eJB_e^T \; - \; m\hat{r}_e\hat{r}_e)\xi \; + \; B_el$$

$$= \; (J_1 \; + \; m|r_e|^2)\xi \; + \; B_el, \tag{7.3.12}$$

$$= \; ((J_1 \; + \; m|r_e|^2)|\xi| \; + \; l_1) \; B_e\mathbf{e_1}.$$

The Lie algebra of the isotropy subgroup G_{μ_e} can be expressed as, cf. (3.1.13), (5.4.44),

$$\mathcal{G}_{\mu_e} \; = \; \{\, \eta \in \mathbb{R}^3 \; : \; \eta \times \mu_e \; = \; 0 \,\},$$

$$= \; \{\, c\,B_e\mathbf{e_1} \; \in \mathbb{R}^3 \; : \; c \in \mathbb{R} \,\}. \tag{7.3.13}$$

The orthogonal complement of $\mathcal{G}_{\mu_e}$ with respect to the locked inertia tensor, as defined in (5.4.18), is

$$
\begin{aligned}
\mathcal{G}_{\mu_e}^{\perp} &= \{\ \zeta \in \mathbb{R}^3\ :\ <\zeta,\ (B_e J B_e^T - m\hat{r}_e\hat{r}_e)\, B_e \mathbf{e}_1 > \ =\ 0\ \}, \\
&= \{\ \zeta \in \mathbb{R}^3\ :\ <\zeta,\ (J_1 + m|r_e|^2)\, B_e \mathbf{e}_1 > \ =\ 0\ \}, \\
&= \{\ c_1 B_e \mathbf{e}_2\ +\ c_2 B_e \mathbf{e}_3\ \in \mathbb{R}^3\ :\ c_1,\ c_2\ \in \mathbb{R}\ \}. \\
&= \mathrm{Span}\ \{\ B_e \mathbf{e}_2,\ B_e \mathbf{e}_3\ \}
\end{aligned}
\tag{7.3.14}
$$

We are now ready to apply the energy-momentum method to study the stability property of the relative equilibrium (7.3.10). Before doing so, we need to find the map $\mathrm{ident}_{\xi}^{Y_o}$ defined in (5.4.49c),

$$
\begin{aligned}
&\mathrm{ident}_{\xi}^{Y_o}(B,\ r)\cdot(\hat{u}B,\ v) \\
&=\ -\big(D\mathbf{I}_{lock}^o(B,\ r)\cdot(\hat{u}B,\ v)\,\xi\big)\ -\ D\mathbf{I}_Y^o(B,\ r)\cdot(\hat{u}B,\ v), \\
&=\ -\hat{u}BJB^T\xi\ +\ \beta J B^T\hat{u}\xi\ +\ m\hat{v}\hat{r}\xi\ +\ m\hat{r}\hat{v}\xi\ -\ \hat{u}Bl.
\end{aligned}
\tag{7.3.15}
$$

Define

$$
\mathbf{u}\ \overset{\triangle}{=}\ B_e^T u,\qquad \mathbf{v}\ \overset{\triangle}{=}\ B_e^T v.
$$

The components of $\mathbf{u}$, $\mathbf{v}$ will be denoted by $(\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3)$ and $(\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3)$, respectively. At relative equilibrium (7.3.10), we write

$$
\begin{aligned}
&\mathrm{ident}_{\xi}^{Y_o}(B_e,\ r_e)\cdot(\hat{u}B_e,\ v) \\
&=\ \mathrm{ident}_{\xi}^{Y_o}(B_e,\ r_e)\cdot(B_e\hat{\mathbf{u}},\ B_e\mathbf{v}) \\
&=\ B_e\big(-J_1|\xi|\hat{\mathbf{u}}\mathbf{e}_1\ +\ |\xi|J\hat{\mathbf{u}}\mathbf{e}_1\ -\ m|\xi|\,|r_e|(\hat{\mathbf{v}}\mathbf{e}_3 + \mathbf{v}_2\mathbf{e}_1)\ -\ l_1\hat{\mathbf{u}}\mathbf{e}_1\big).
\end{aligned}
\tag{7.3.16}
$$

The block diagonalization technique is based on the decomposition of the space $\mathcal{V}$ defined in (5.4.29). For the relative equilibrium $(B_e,\ r_e)$ satisfying (7.3.10), we check, since $\mathcal{G}_{\mu_e} = \mathrm{Span}\{B_e\mathbf{e}_1\}$,

$$
\begin{aligned}
\ll (\hat{u}B_e,\ v),\ (B_e\widehat{\mathbf{e}_1},\ \widehat{B_e\mathbf{e}_1}\, r_e) \gg\ &=\ <B_e^T u,\ \mathbf{J}\mathbf{e}_1> +m|r_e| < v,\ \widehat{B_e\mathbf{e}_1}\, B_e\mathbf{e}_2>, \\
&=\ J_1\mathbf{u}_1\ +\ m|r_e|\mathbf{v}_3.
\end{aligned}
$$

Thus the space $\mathcal{V}$ can be represented as

$$
\mathcal{V}\ =\ \{\ (B_e\hat{\mathbf{u}},\ B_e\mathbf{v})\ :\ J_1\mathbf{u}_1\ +\ m|r_e|\mathbf{v}_3\ =\ 0\ \}.
\tag{7.3.17}
$$

It has the subspace, cf. (5.4.31),

$$\mathcal{V}_{RIG} = \{ (\hat{\zeta}B_e, \ \hat{\zeta}r_e) \ : \ \zeta \in \mathcal{G}_{\mu_e}^{\perp} \ \}$$

$$= \text{Span} \ \{ \ (B_e\widehat{\mathbf{e_2}}, \ 0), \ (B_e\widehat{\mathbf{e_3}}, \ -|r_e|B_e\mathbf{e_1}) \ \}. \qquad (7.3.18)$$

From (7.3.16), we may find the other subspace $\mathcal{V}_{INT}$, defined in (5.4.32), as follows. In order to have

$$< B_e\mathbf{e_2}, \ B_e\big(-J_1|\xi|\hat{\mathbf{u}}\mathbf{e_1} \ + \ |\xi|\mathbf{J}\hat{\mathbf{u}}\mathbf{e_1} \ - \ m|\xi|\,|r|(\hat{\mathbf{v}}\mathbf{e_3} + \mathbf{v_2}\mathbf{e_1}) \ - \ l_1\hat{\mathbf{u}}\mathbf{e_1}\big) > \ = 0,$$

$$< B_e\mathbf{e_3}, \ B_e\big(-J_1|\xi|\hat{\mathbf{u}}\mathbf{e_1} \ + \ |\xi|\mathbf{J}\hat{\mathbf{u}}\mathbf{e_1} \ - \ m|\xi|\,|r|(\hat{\mathbf{v}}\mathbf{e_3} + \mathbf{v_2}\mathbf{e_1}) \ - \ l_1\hat{\mathbf{u}}\mathbf{e_1}\big) > \ = 0,$$

we get the conditions

$$\big((J_1 - J_2)|\xi| \ + \ l_1\big)\mathbf{u_3} \ = \ m|\xi|\,|r_e|\mathbf{v_1}, \qquad (7.3.19a)$$

$$\big((J_1 - J_3)|\xi| \ + \ l_1\big)\mathbf{u_2} \ = \ 0. \qquad (7.3.19b)$$

Assuming $(J_1 - J_3)|\xi| + l_1 \neq 0$, we get the representation of $\mathcal{V}_{INT}$,

$$\mathcal{V}_{INT} \ = \ \{ \ (B_e\hat{\mathbf{u}}, \ B_e\mathbf{v}) \ : J_1\mathbf{u_1} \ + \ m|r_e|\mathbf{v_3} \ = \ 0,$$

$$\big((J_1 - J_2)|\xi| \ + \ l_1\big)\mathbf{u_3} \ = \ m|\xi|\,|r_e|\mathbf{v_1}, \qquad (7.3.20)$$

$$\mathbf{u_2} \ = \ 0 \ \}$$

Next we check the conditions in Theorem 5.4.8. First, since for this example $G = SO(3)$, as in the previous example, checking the Arnold block becomes checking the positive definiteness of $\mathbf{I}_{lock}(B_e, r_e)^{-1} \ -- \ \frac{1}{\lambda}\mathbf{1}$ in any direction except ξ. At relative equilibrium (7.3.10), we may express the locked inertia dyadic as

$$\mathbf{I}_{lock}^o(B_e, \ r_e) \ = \ B_e\mathcal{J}B_e^T, \qquad (7.3.21a)$$

where

$$\mathcal{J} \ = \ \begin{pmatrix} J_1 + m|r_e|^2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 + m|r_e|^2 \end{pmatrix}. \qquad (7.3.21b)$$

The scaler λ can be found through (7.3.12),

$$\lambda \ = \ \frac{(J_1 \ + \ m|r_e|^2)\,|\xi| \ + \ l_1}{|\xi|}. \qquad (7.3.22)$$

Consequently, we may express

161

$$\mathbf{I}_{lock}(B_e, \ r_e)^{-1} - \frac{1}{\lambda}\mathbf{1} \ = \ B_e \begin{pmatrix} \frac{1}{J_1 + m|r_e|^2} - \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{J_2} - \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{J_3 + m|r_e|^2} - \frac{1}{\lambda} \end{pmatrix} B_e^T.$$

$$(7.3.23)$$

By taking out the direction ξ, which is along $B_e \mathbf{e_1}$, positive definiteness of the Arnold block is ensured by the conditions

$$\frac{1}{J_2} - \frac{1}{\lambda} > 0, \quad \text{and} \quad \frac{1}{J_3 + m|r_e|^2} - \frac{1}{\lambda} > 0.$$

By substituting λ in (7.3.22) into these conditions, we get

$$J_1 \ + \ m|r_e|^2 \ + \ \frac{l_1}{|\xi|} \ > \ J_2,$$

$$J_1 \ + \ m|r_e|^2 \ + \ \frac{l_1}{|\xi|} \ > \ J_3 \ + \ m|r_e|^2.$$

For the case that $|r_e|$ sufficiently large, the first condition is trivially satisfied. Accordingly, for positive definiteness of the Arnold block, one asks

$$l_1 \ > \ (J_3 \ - \ J_1) \ |\xi|. \qquad\qquad (7.3.24)$$

Recall from Lemma 5.4.7, condition (7.3.24) implies the decomposition of $\mathcal{V}$, i.e. (5.4.33) holds.

Now we check the other block, namely, $\mathcal{V}_{INT} \times \mathcal{V}_{INT}$. The second variation of V_ξ at relative equilibrium $(B_e, \ r_e)$ satisfying (7.3.20) can be found as follows.

$$D^2 V_\xi(B, \ r) \cdot (\hat{u}_1 B, \ v_1) \cdot (\hat{u}_2 B, \ v_2)$$

$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} DV_\xi(e^{\epsilon \hat{u}_2} B, \ r + \epsilon v_2) \cdot (\hat{u}_1 e^{\epsilon \hat{u}_2} B, \ v_1)$$

$$= \ < \frac{GM}{|r|^3}\left(m + \frac{3}{2|r|^2}\, tr(\mathbf{J}) - \frac{15}{2|r|^4}\, r^T B J B^T r\right)\ v_2$$

$$- \frac{3GM}{|r|^5}\left(m + \frac{5}{2|r|^2}\, tr(\mathbf{J}) - \frac{105}{2|r|^4}\, r^T B J B^T r\right)\ r\, r^T v_2$$

$$- \frac{15GM}{|r|^7}\left(v_2^T B J B^T r + r^T \hat{u}_2 B J B^T r\right)\ r$$

$$+ \frac{3GM}{|r|^5}(\hat{u}_2 B J B^T r - B J B^T \hat{u}_2 r + B J B^T v_2)$$

$$+ m\hat{\xi}\hat{\xi}v_2 - \frac{15GM}{|r|^7}B J B^T r r^T v_2, \ v_1 >$$

$$+ \ < -\frac{15GM}{|r|^7}\hat{r} B J B^T r r^T v_2$$

$$+ \frac{3GM}{|r|^5}\left(-\hat{v}_2 B J B^T r - \hat{r}\hat{u}_2 B J B^T r + \hat{r} B J B^T \hat{u}_2 r - \hat{r} B J B^T v_2\right)$$

$$- \hat{\xi} B J B^T \hat{u}_2 \xi + \hat{\xi}\hat{u}_2 (B J B^T \xi + Bl), \ u_1 > .$$

At relative equilibrium (B_e, r_e) satisfying (7.3.10), we can express $D^2 V_\xi$ in terms of a quadratic form,

$$D^2 V_\xi(B_e, \ r_e) \cdot (B_e\hat{u}, \ B_e\mathbf{v}) \cdot (B_e\hat{u}, \ B_e\mathbf{v})$$

$$= \left(m|\xi|^2 + \frac{3GM}{|r_e|^5}(J_1 - J_2)\right)\ \mathbf{v}_1^2 + \left(-5m|\xi|^2 + \frac{2mGM}{|r_e|^3}\right)\ \mathbf{v}_2^2$$

$$+ \frac{3GM}{|r_e|^5}(J_3 - J_2)\ \mathbf{v}_3^2 + \frac{6GM}{|r_e|^4}(J_1 - J_3)\ u_1\mathbf{v}_3 + \frac{3GM}{|r_e|^3}(J_3 - J_2)\ u_1^2$$

$$+ \left(|\xi|^2(J_1 - J_3) + l_1|\xi|\right)\ u_2^2 + \left((|\xi|^2 + \frac{3GM}{|r_e|^3})(J_1 - J_2) + l_1|\xi|\right)\ u_3^2$$

For $(B_e\hat{u}, \ B_e\mathbf{v}) \in \mathcal{V}_{INT}$, we can further simplify the above formula by using (7.3.20) as

$$D^2 V_\xi(B_e, \ r_e) \cdot \delta x \cdot \delta x$$

$$= \left\{m|\xi|^2 + \frac{3GM}{|r_e|^5}(J_1 - J_2) + \left(\frac{m|\xi|\,|r_e|}{(J_1 - J_2)|\xi| + l_1}\right)^2\right.$$

$$\left.\left((|\xi|^2 + \frac{3GM}{|r_e|^3})(J_1 - J_2) + l_1|\xi|\right)\right\}\ \mathbf{v}_1^2 \qquad (7.3.25)$$

$$+ \left(-5m|\xi|^2 + \frac{2mGM}{|r_e|^3}\right)\ \mathbf{v}_2^2$$

$$+ \left(\frac{3GM}{|r_e|^5}(J_3 - J_2) - \frac{6mGM}{J_1|r_e|^3}(J_1 - J_3) + \frac{3m^2GM}{J_1^2|r_e|}(J_3 - J_2)\right)\ \mathbf{v}_3^2.$$

Since the coefficient of v_2^2 is, by applying (7.3.11),

$$- 5m|\xi|^2 + \frac{2mGM}{|r_e|^3}$$

$$= -\frac{5GM}{|r_e|^3}\left(m + \frac{3}{2|r_e|^2}\,tr(\mathbf{J}) - \frac{9}{2|r_e|^2}\,J_2\right) + \frac{2mGM}{|r_e|^3}$$

$$= -\frac{3mGM}{|r_e|^3} - \frac{15GM}{2|r_e|^5}(J_1 + J_3 - 2J_2),$$

which is always negative. Thus the second variation of V_ξ on $\mathcal{V}_{INT} \times \mathcal{V}_{INT}$ will never be positive definite. Consequently, for this example, V_ξ is *insufficient* for stability. We actually need to use the full power of energy-momentum method, namely, checking the bilinear form $\mathbf{B}_\xi(x_e)$ defined in (5.4.34), or the amended potential V_μ, cf. (5.4.43).

We next find the other term in (5.4.34). From (7.3.16), (7.3.21), we get

$$\mathtt{ident}_\xi^{Y,o}(B_e,\ r_e)(B_e\hat{\mathbf{u}},\ B_e\mathbf{v}) \cdot \mathbf{I}_{lock}^o(B_e,\ r_e)^{-1}\mathtt{ident}_\xi^{Y,o}(B_e,\ r_e)(B_e\hat{\mathbf{u}},\ B_e\mathbf{v})$$

$$= \ <\ |\xi|(-J_1\mathbf{1} + \mathbf{J})\hat{\mathbf{u}}\mathbf{e_1}\ -m|\xi|\,|r_e|(\hat{v}\mathbf{e_3} + v_2\mathbf{e_3})\ -\ l_1\hat{\mathbf{u}}\mathbf{e_1},$$

$$\mathcal{J}^{-1}\big(|\xi|(-J_1\mathbf{1} + \mathbf{J})\hat{\mathbf{u}}\mathbf{e_1}\ -m|\xi|\,|r_e|(\hat{v}\mathbf{e_3} + v_2\mathbf{e_3})\ -\ l_1\hat{\mathbf{u}}\mathbf{e_1}\big)\ >,$$

where $\mathcal{J}$ is defined in (7.3.21b). With further calculations, we can express this bilinear form on the space $\mathcal{V}_{INT} \times \mathcal{V}_{INT}$ as

$$\mathtt{ident}_\xi^{Y,o}(x_e)\delta x \cdot \mathbf{I}_{lock}^o(x_e)^{-1}\mathtt{ident}_\xi^{Y,o}(x_e)\delta x\bigg|_{\mathcal{V}_{INT} \times \mathcal{V}_{INT}} = \frac{4m^2|\xi|^2|r_e|^2}{J_1 + m|r_e|^2}\,v_2^2 \qquad (7.3.26)$$

By combining (7.3.25), (7.3.26), we end up with the bilinear form $\mathbf{B}_\xi(x_e)$,

$$\mathbf{B}_\xi(x_e)\,(\delta x,\ \delta x)\bigg|_{\mathcal{V}_{INT} \times \mathcal{V}_{INT}}$$

$$= \left\{m|\xi|^2 + \frac{3GM}{|r_e|^5}(J_1 - J_2) + \left(\frac{m|\xi|\,|r_e|}{(J_1 - J_2)|\xi| + l_1}\right)^2 \right.$$

$$\left. \left((|\xi|^2 + \frac{3GM}{|r_e|^3})(J_1 - J_2) + l_1|\xi|\right)\right\}\,v_1^2 \qquad (7.3.27)$$

$$+ \left(-5m|\xi|^2 + \frac{2mGM}{|r_e|^3} + \frac{4m^2|\xi|^2|r_e|^2}{J_1 + m|r_e|^2}\right)\,v_2^2$$

$$+ \left(\frac{3GM}{|r_e|^5}(J_3 - J_2) - \frac{6mGM}{J_1|r_e|^3}(J_1 - J_3) + \frac{3m^2GM}{J_1^2|r_e|}(J_3 - J_2)\right)\,v_3^2.$$

Now the coefficient for the term v_2^2 has been changed. We can write it now as,

$$-5m|\xi|^2 + \frac{2mGM}{|r_e|^3} + \frac{4m^2|\xi|^2|r_e|^2}{J_1 + m|r_e|^2}$$

$$= \left(1 - \frac{4J_1}{J_1 + m|r_e|^2}\right)\frac{mGM}{|r_e|^3} - \left(1 + \frac{4J_1}{J_1 + m|r_e|^2}\right)\frac{3GM}{2|r_e|^5}(J_1 + J_3 - 2J_2).$$

It is clear then for $|r_e|$ sufficiently large, the first term dominates with coefficient

$$1 - \frac{4J_1}{J_1 + m|r_e|^2} > 0.$$

Also for $|r_e|$ large, the third term in the coefficient of v_3^2 in (7.3.27) dominates. Consequently, the quadratic form in (7.3.27) can be made positive definite with the conditions

$$J_1 > J_2, \quad \text{and} \quad J_3 > J_2. \tag{7.3.28}$$

Combining conditions in (7.3.24), (7.3.28), we state the sufficient conditions for stability in the following theorem.

THEOREM 7.3.1

For the problem of a rigid body with momentum wheel in a central gravitational force field, conditions

$$\xi = |\xi| \, B_e \mathbf{e}_1,$$

$$l = l_1 \mathbf{e}_1, \quad \text{with} \quad l_1 \geq 0,$$

$$r_e = |r_e| \, B_e \mathbf{e}_2.$$

give rise to a relative equilibrium $(B_e, \ r_e)$. If, furthermore, for $|r_e|$ sufficiently large, we have

$$l_1 > (J_3 - J_1) \, |\xi|,$$

$$J_1 > J_2,$$

$$J_3 > J_2,$$

then this relative equilibrium is relatively stable. ∎

REMARK 7.3.2

In particular, for the case that $l = 0$, the sufficient conditions become

$$J_1 > J_3, \quad J_1 > J_2, \quad \text{and} \quad J_3 > J_2,$$

which is equivalent to the condition,

$$J_1 > J_3 > J_2.$$

This is exactly the same condition we obtained from the other two methods in Chapter 6, namely, the energy-Casimir method and Lagrange-multiplier method, cf. Theorem 6.5.2.

∎

7.4. Multibody Dual-Spin Problem

In this section, we will continue our study of the multibody analog of dual-spin problem discussed in Sections 4.3, 4.4. Recall that in Theorem 4.4.1, with appropriate damping mechanism, the system depicted in Figure 4.3.1 asymptotically approaches one of the stable relative equilibrium corresponding to a suitable gyroscopic system with symmetry. Here we will compute a stable relative equilibrium. From the construction, this gyroscopic system has the following entities, cf. (3.5.5a),

$$Q = SO(3) \times SO(3),$$

$$K\left((B\hat{u}_1,\ B\hat{u}_2),\ (B\hat{w}_1,\ B\hat{w}_2)\right) = \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \begin{pmatrix} \mathbf{J}_1 & \mathbf{J}_{12} \\ \mathbf{J}_{12}^T & \mathbf{J}_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$Y(B_1,\ B_2) = (B_1\hat{y}_1,\ B_2\hat{y}_2), \tag{7.4.1}$$

$$V(B) = 0,$$

$$G = SO(3),$$

where, cf. (3.5.5b), (4.3.6b),

$$\mathbf{J}_1 = \mathbf{I}_1 + \varepsilon\, \hat{d}_1^T \hat{d}_1 + \sum_{i=1}^{3} \mathbf{I}_{S_i},$$

$$\mathbf{J}_2 = \mathbf{I}_2 + \varepsilon\, \hat{d}_2^T \hat{d}_2 + \sum_{i=1}^{3} \mathbf{I}_{D_i}, \tag{7.4.2}$$

$$\mathbf{J}_{12} = \varepsilon\, \hat{d}_1 B_1^T B_2 \hat{d}_2,$$

and the components y_1, y_2 of the gyroscopic field are given by solving the equations

$$J_1 \, y_1 \; + \; J_{12} \, y_2 \; = \; l \; = \; I^S \dot{\Theta},$$

$$J_{12}^T \, y_1 \; + \; J_2 \, y_2 \; = \; 0.$$

(7.4.3)

The Lagrangian for this system can be written as

$$L(B_1, \Omega_1, B_2, \Omega_2) \; = \; \frac{1}{2} < \Omega_1, \, J_1 \Omega_1 > + \frac{1}{2} < \Omega_2, \, J_2 \Omega_2 >$$
$$+ \, \varepsilon \, < \Omega_1, \, \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2 > \, + \, < \Omega_1, \, l > .$$

(7.4.4)

The locked inertia dyadic can be found through

$$< \xi, \, I_{lock}^o(B_1, B_2) \eta >$$

$$= \; \ll \hat{\xi}_Q(B_1, B_2), \, \hat{\eta}_Q(B_1, B_2) \gg$$

$$= \; < \xi, \, (B_1 J_1 B_1^T + B_2 J_2 B_2^T + B_1 J_{12} B_2^T + B_2 J_{12}^T B_1^T) \eta > .$$

Thus we have

$$I_{lock}^o(B_1, B_2) \; = \; B_1 J_1 B_1^T + B_2 J_2 B_2^T + B_1 J_{12} B_2^T + B_2 J_{12}^T B_1^T.$$

(7.4.5)

The gyro-momentum in $\mathcal{G}^*$ induced by the gyroscopic field Y can be found from definition (5.4.15), see also (5.4.48), to be

$$I_Y^o \; = \; B_1 \, l.$$

(7.4.6)

Accordingly, the momentum mapping is

$$\mu \; = \; I_{lock}^o(x) \xi \; + \; I_Y^o(x)$$

$$= \; (B_1 J_1 B_1^T + B_2 J_2 B_2^T + B_1 J_{12} B_2^T + B_2 J_{12}^T B_1^T) \xi \; + \; B_1 l.$$

(7.4.7)

The augmented potential function V_ξ is

$$V_\xi(B_1, B_2) \; = \; -\frac{1}{2} < \xi, \, I_{lock}^c \xi > \; - \; < B_1 l, \, \xi >,$$

$$= \; -\frac{1}{2} < \xi, \, (B_1 J_1 B_1^T + B_2 J_2 B_2^T + B_1 J_{12} B_2^T + B_2 J_{12}^T B_1^T) \xi >$$

$$- \; < \xi, \, B_1 l >,$$

$$= \; -\frac{1}{2} < \xi, \, (B_1 J_1 B_1^T + B_2 J_2 B_2^T + \varepsilon \, \widehat{B_1 d_1} \widehat{B_2 d_2} + \varepsilon \, \widehat{B_2 d_2} \widehat{B_1 d_1}) \xi >$$

$$- \; < \xi, \, B_1 l >,$$

(7.4.8)

Now we apply the Principle of Symmetric Criticality to find the conditions for relative equilibrium. The first variation of the augmented potential can be derived as follows,

167

$$DV_\xi(B_1, \ B_2) \cdot (\hat{u}_1 B_1, \ \hat{u}_2 B_2)$$

$$= < \hat{\xi} B_1 \mathbf{J}_1 B_1^T \xi \ + \ \hat{\xi} B_1 l \ + \ \varepsilon \widehat{B_1 d_1} \hat{\xi} \widehat{B_2 d_2} \xi, \ u_1 > \qquad (7.4.9)$$

$$+ \ < \hat{\xi} B_2 \mathbf{J}_2 B_2^T \xi \ + \ \varepsilon \widehat{B_2 d_2} \hat{\xi} \widehat{B_1 d_1} \xi, \ u_2 > \ .$$

From the above formula, we read out the conditions for the configuration components of the relative equilibrium (B_{1e}, B_{2e}) as satisfying

$$\xi \times (B_1 \mathbf{J}_1 B_1^T \xi + B_1 l) \ + \ \varepsilon \, B_1 d_1 \times \left(\xi \times (B_2 d_2 \times \xi) \right) \ = \ 0, \qquad (7.4.10a)$$

$$\xi \times (B_2 \mathbf{J}_2 B_2^T \xi) \ + \ \varepsilon \, B_2 d_2 \times \left(\xi \times (B_1 d_1 \times \xi) \right) \ = \ 0, \qquad (7.4.10b)$$

These are very similar to the conditions we derived before, cf. (3.5.9), except that the gyroscopic term enters. By taking dot product with ξ on both side of (7.4.10a), and letting $s_1 = B_1 d_1$, $s_2 = B_2 d_2$, cf. (3.5.11), we obtain again the coplanarity condition,

$$\xi \ \cdot \ (s_1 \ \times \ s_2) \ = \ 0. \qquad (7.4.11)$$

Consequently, the gyroscopic term doesn't affect the coplanarity condition for the relative equilibrium for this problem. With this coplanar condition, conditions (7.4.10) may be re-expressed as

$$\xi \times (B_1 \mathbf{J}_1 B_1^T \xi + B_1 l) \ - \ \varepsilon \, (B_1 d_1 \cdot \xi)(B_2 d_2 \times \xi) \ = \ 0, \qquad (7.4.12a)$$

$$\xi \times (B_2 \mathbf{J}_2 B_2^T \xi) \ - \ \varepsilon \, (B_2 d_2 \cdot \xi)(B_1 d_1 \times \xi) \ = \ 0. \qquad (7.4.12b)$$

Now we find a particular relative equilibrium for this problem. Let $\{e_1, e_2, e_3\}$, $\{f_1, f_2, f_3\}$ be the coordinate frames corresponding to body 1, body 2, respectively, such that

$$\mathbf{J}_1 \, e_i \ = \ \mathbf{J}_{1i} \, e_i, \qquad \mathbf{J}_2 \, f_i \ = \ \mathbf{J}_{2i} \, f_i, \quad i = 1, 2, 3.$$

It can be checked that if the following conditions holds,

$$\xi \ = \ |\xi| \ B_1 \, e_1 \ = \ |\xi| \ B_2 \, f_1, \qquad (7.4.13a)$$

$$l \ = \ l_1 \, e_1, \qquad (7.4.13b)$$

$$d_1 \ = \ a_1 \, e_2, \qquad d_2 \ = \ a_2 \, f_2, \qquad (7.4.13c)$$

168

conditions (7.4.12) are satisfied. Thus these conditions (7.4.13) give rise to a relative equilibrium (B_{1e}, B_{2e}). From (7.4.13a), we know that

$$B_{1e}\ \mathbf{e_1}\ =\ B_{2e}\ \mathbf{f_1}. \tag{7.4.14}$$

By substituting (7.4.13a), (7.4.13c) in the coplanarity condition (7.4.11), we get

$$\mathbf{e_1} \cdot (\mathbf{e_2} \times B_e \mathbf{f_2})\ =\ 0, \tag{7.4.15}$$

where $B_e = B_{1e}^T B_{2e}$. With (7.4.14), this only happens when

$$B_e\ \mathbf{f_2}\ =\ \pm\ \mathbf{e_2}.$$

Accordingly, we have two sets of relative equilibria expressed in terms of the relative shape variable B_e,

$$B_e\ \mathbf{f_1}\ =\ \mathbf{e_1}, \quad B_e\ \mathbf{f_2}\ =\ \mathbf{e_2}, \quad B_e\ \mathbf{f_3}\ =\ \mathbf{e_3}, \tag{7.4.16a}$$

$$B_e\ \mathbf{f_1}\ =\ \mathbf{e_1}, \quad B_e\ \mathbf{f_2}\ =\ -\mathbf{e_2}, \quad B_e\ \mathbf{f_3}\ =\ -\mathbf{e_3}. \tag{7.4.16b}$$

In the following, we will study the stability property of the relative equilibrium corresponding to (7.4.16b) with the conditions (7.4.13). This configuration may be depicted in Figure 7.4.1.
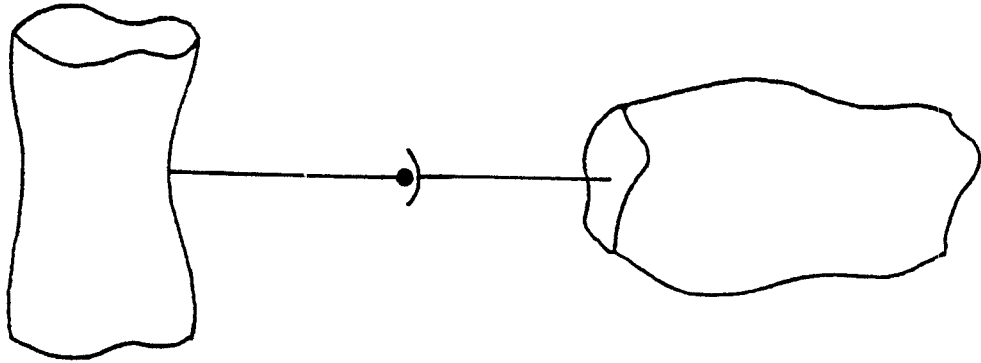

Figure 7.4.1. Relative Equilibrium Configuration.

The energy-momentum method will be adopted here to determine the stability conditions. We first need to compute the second variation of the augmented potential. It can be found as follows,

$$
\begin{aligned}
D^2 V_\xi(B_1, B_2) &\cdot (\hat{u}_1 B_1, \hat{u}_2 B_2) \cdot (\hat{u}_1 B_1, \hat{u}_2 B_2) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} DV_\xi(e^{\epsilon \hat{u}_1} B_1, e^{\epsilon \hat{u}_2} B_2) \cdot (\hat{u}_1 e^{\epsilon \hat{u}_1} B_1, \hat{u}_2 e^{\epsilon \hat{u}_2} B_2) \\
&= \; < \hat{u}_1 B_1 J_1 B_1^T \xi - B_1 J_1 B_1^T \hat{u}_1 \xi + \hat{u}_1 B_1 l, \; \hat{u}_1 \xi > \\
&\quad + \; < \hat{u}_2 B_2 J_2 B_2^T \xi - B_2 J_2 B_2^T \hat{u}_2 \xi, \; \hat{u}_2 \xi > \\
&\quad + \; \varepsilon < (\widehat{\hat{u}_1 B_1 d_1})\hat{\xi} \widehat{B_2 d_2} \xi, \; u_1 > \; + \; \varepsilon < (\widehat{\hat{u}_2 B_2 d_2})\hat{\xi} \widehat{B_1 d_1} \xi, \; u_2 > \\
&\quad + \; 2\varepsilon < (\widehat{\hat{u}_1 B_1 d_1})\xi, \; (\widehat{\hat{u}_2 B_2 d_2})\xi > \; .
\end{aligned}
\tag{7.4.17}
$$

Define

$$
\mathbf{u_1} \triangleq B_{1e}^T u_1, \qquad \mathbf{u_2} \triangleq B_{2e}^T u_2,
\tag{7.4.18}
$$

The components of $\mathbf{u_1}$, $\mathbf{u_2}$ will be denoted by $(\mathbf{u}_{11}, \mathbf{u}_{12}, \mathbf{u}_{13})$ and $(\mathbf{u}_{21}, \mathbf{u}_{22}, \mathbf{u}_{23})$, respectively. Also, we will use the notations,

$$
\mathbf{J}_1 = \text{diag}\{ J_{11}, J_{12}, J_{13} \}, \qquad \mathbf{J}_2 = \text{diag}\{ J_{21}, J_{22}, J_{23} \}.
\tag{7.4.19}
$$

At relative equilibrium (B_{1e}, B_{2e}) such that (7.4.16b), (7.4.13) hold, and we can further write the second variation of the augmented potential as

$$
\begin{aligned}
D^2 V_\xi(B_{1e}, B_{2e}) &\cdot (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}) \cdot (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}) \\
&= \; \varepsilon |\xi|^2 a_1 a_2 (\mathbf{u}_{11} - \mathbf{u}_{21})^2 \; + \; ((J_{11} - J_{13})|\xi| + l_1)|\xi|\mathbf{u}_{12}^2 \\
&\quad + \; ((J_{11} - J_{12} + \varepsilon a_1 a_2)|\xi| + l_1)|\xi|\mathbf{u}_{13}^2 \; + \; (J_{21} - J_{23})|\xi|^2 \mathbf{u}_{22}^2 \\
&\quad + \; (J_{21} - J_{22} + \varepsilon a_1 a_2)|\xi|^2 \mathbf{u}_{23}^2 .
\end{aligned}
\tag{7.4.20}
$$

From Theorem 5.4.10, we check the positive definiteness of the second variation of the augmented potential on the space of $\mathcal{V}$. For the relative equilibrium under investigation, we have the momentum mapping, cf. (7.4.7),

$$
\mu_e = \left((J_{11} + J_{21} + 2\varepsilon a_1 a_2)|\xi| + l_1 \right) B_{1e} \, \mathbf{e_1}.
\tag{7.4.21}
$$

Thus the Lie algebra corresponding to the isotropy group is

$$
\mathcal{G}_{\mu_e} = \text{Span}\{ B_{1e} \, \mathbf{e_1} \},
\tag{7.4.22}
$$

170

with the orthogonal complement with respect to the locked inertia tensor,

$$\mathcal{G}^{\perp}_{\mu_e} = \text{Span}\{ \, B_{1e} \, \mathbf{e_2}, \, B_{1e} \, \mathbf{e_3} \, \}. \qquad (7.4.23)$$

The space $\mathcal{V}$ can then be found through

$$\mathcal{V} = \{ \, (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}) : \, \ll (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}), \, (\hat{\eta}B_{1e}, \, \hat{\eta}B_{2e}) \gg \, = \, 0, \, \forall \, \eta \in \mathcal{G}_{\mu_e} \, \},$$

$$= \{ \, (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}) : \, (J_{11} + \varepsilon \, a_1 a_2)\mathbf{u}_{11} \, + \, (J_{21} + \varepsilon \, a_1 a_2)\mathbf{u}_{21} \, = \, 0 \, \}. \qquad (7.4.24)$$

The second variation of the augmented potential restricted to $\mathcal{V}$ can then be written as

$$\left. D^2 V_\xi(B_{1e}, B_{2e}) \right|_{\mathcal{V} \times \mathcal{V}} \cdot (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}}) \cdot (B_{1e}\widehat{\mathbf{u_1}}, B_{2e}\widehat{\mathbf{u_2}})$$

$$= \varepsilon \, |\xi|^2 a_1 \, a_2 \left(\frac{J_{21} + \varepsilon \, a_1 a_2}{J_{11} + \varepsilon \, a_1 a_2} + 1 \right)^2 \mathbf{u}_{21}^2 \qquad (7.4.25)$$

$$+ \, ((J_{11} - J_{13})|\xi| + l_1)|\xi|\mathbf{u}_{12}^2 \, + \, ((J_{11} - J_{12} + \varepsilon \, a_1 a_2)|\xi| + l_1)|\xi|\mathbf{u}_{13}^2$$

$$+ \, (J_{21} - J_{23})|\xi|^2\mathbf{u}_{22}^2 \, + \, (J_{21} - J_{22} + \varepsilon \, a_1 a_2)|\xi|^2\mathbf{u}_{23}^2.$$

Consequently, we can read off the sufficient conditions for stability from (7.4.25) as,

$$(J_{11} - J_{13})|\xi| \, + \, l_1 \, > \, 0, \qquad (7.4.26a)$$

$$(J_{11} - J_{12} + \varepsilon \, a_1 a_2)|\xi| \, + \, l_1 \, > \, 0, \qquad (7.4.26b)$$

$$J_{21} - J_{23} \, > \, 0, \qquad (7.4.26c)$$

$$J_{21} - J_{22} + \varepsilon \, a_1 a_2 \, > \, 0, \qquad (7.4.26d)$$

The above discussions was summarized in the following theorem.

<u>THEOREM 7.4.1</u>

For the multibody dual-spin problem, conditions (7.4.13), (7.4.16b) give rise to a relative equilibrium (B_{1e}, B_{2e}). Furthermore, assuming that

$$\frac{l_1}{|\xi|} \, > \, J_{13} \, - \, J_{11}, \qquad (7.4.27a)$$

$$\frac{l_1}{|\xi|} \, > \, J_{12} \, - \, J_{11} \, - \, \varepsilon \, a_1 a_2, \qquad (7.4.27b)$$

the relative equilibrium (B_{1e}, B_{2e}) is stable if

$$J_{21} - J_{23} > 0, \qquad\qquad (7.4.28a)$$

$$J_{21} - J_{22} + \varepsilon\, a_1 a_2 > 0. \qquad\qquad (7.4.28b)$$

∎

REMARK 7.4.2

It may be checked that the positive definiteness conditions for the Arnold block are

$$(J_{11} - J_{13} + J_{21} - J_{23})|\xi| + l_1 > 0, \qquad (7.4.29a)$$

$$(J_{11} - J_{12} + J_{21} - J_{22} + 2\varepsilon\, a_1 a_2)|\xi| + l_1 > 0. \qquad (7.4.29b)$$

It is easy to see that these conditions are implied by the conditions in Theorem 7.3.1. However, this is not enough for stability. There are additional conditions coming from the other block. Thus, *for such a coupled system, we could never regard the system as a whole rigid body*. The coupling effects should be suitably accommodated.

∎

Now we consider the other relative equilibrium coming from (7.4.16a). The second variation of the augmented potential corresponding to the case that the relative shape is identity, or the two bodies are *folded* can be found from (7.4.17) to be, cf. (7.4.20),

$$D^2 V_\xi(B_{1e}, B_{2e}) \cdot (B_{1e}\widehat{u_1}, B_{2e}\widehat{u_2}) \cdot (B_{1e}\widehat{u_1}, B_{2e}\widehat{u_2})$$

$$= -\varepsilon\, |\xi|^2 a_1\, a_2 (u_{11} - u_{21})^2 + ((J_{11} - J_{13})|\xi| + l_1)|\xi|u_{12}^2$$

$$+ ((J_{11} - J_{12} - \varepsilon\, a_1 a_2)|\xi| + l_1)|\xi|u_{13}^2 + (J_{21} - J_{23})|\xi|^2 u_{22}^2$$

$$+ (J_{21} - J_{22} - \varepsilon\, a_1 a_2)|\xi|^2 u_{23}^2. \qquad\qquad (7.4.30)$$

Even restricted to the space $\mathcal{V}$, there is always one negative term. This fact suggests that this relative equilibrium may be unstable, irrespective of the rotor speed. Further analysis is needed to justify this.

7.5. Rigid Body with a Flexible Attachment

In this section, we consider the system of a rigid body attached with a string, or a shear beam. We shall first establish the model for this system and then study stability properties. The methodology introduced in Chapter 2 will be extended to the infinite dimensional case. Similar problems are discussed in [55], [38].

Consider the system configuration depicted in Figure 7.5.1. Let $\mathcal{M}$ denote the space of smooth functions from the interval $[0, L]$ to $\mathbb{R}^3$. Let $B \in SO(3)$ be the attitude of the rigid body, with associated coordinate system $\{e_1, e_2, e_3\}$. The relative angles between the rigid body and the rotors are denoted by θ_1, θ_2, θ_3, respectively. With $\sigma \in \mathcal{M}$, the configuration space of the system can be expressed as

$$
\begin{aligned}
Q &:= SO(3) \times (S^1)^3 \times \mathcal{M} \\
&:= \{ \, x = (B, \, \theta_1, \, \theta_2, \, \theta_3, \, \sigma) \, \}.
\end{aligned}
\tag{7.5.1}
$$

This may be alternatively written formally as a manifold, cf. (4.3.1),

$$
\begin{aligned}
Q_u &= SO(3) \times (SO(3))^3 \times \mathcal{M} \\
&= \{ \, (B, \, S_1, \, S_2, \, S_3, \, \sigma) \, \}.
\end{aligned}
\tag{7.5.2}
$$

with kinematic constraints, cf. (4.3.2a), (4.3.3),

$$
S_i = B \, R(e_i, \theta_i), \quad i = 1, 2, 3.
\tag{7.5.3}
$$

Note that eventually we need to be precise about appropriate Hilbert manifold structures on Q, TQ, etc. In this dissertation, we limit ourselves to the formal aspects only. The tangent space TQ of Q can be written as

$$
TQ = \{ \, v_x = (B, \, \theta_1, \, \theta_2, \, \theta_3, \, \sigma, \, B\hat{\Omega}, \, \omega_1, \, \omega_2, \, \omega_3, \, \nu) \, \}.
\tag{7.5.4}
$$

By using the techniques developed in Section 2.2, we may write the tangent space to TQ (second tangent space) and its dual as

$$
T_{v_x} TQ = \{ \, \overline{v_x} = (B\hat{u}, \, \zeta_1, \, \zeta_2, \, \zeta_3, \, \bar{\sigma}, \, B(\hat{u}\hat{\Omega} + \hat{y}), \, \eta_1, \, \eta_2, \, \eta_3, \, \bar{\nu}) \},
\tag{7.5.5}
$$

$$
T_{v_x}^* TQ = \{ \, v_x^* = (B(\dot{\hat{\beta}}\hat{\Omega} + \hat{\alpha}), \, a_1, \, a_2, \, a_3, \, \sigma^*, \, B\hat{\beta}, \, b_1, \, b_2, \, b_3, \, \nu^*) \},
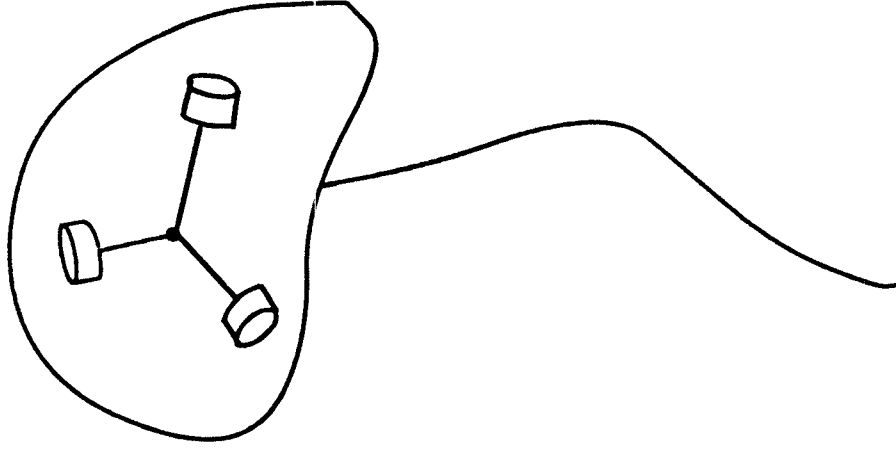\tag{7.5.6}
$$

with the pairing,

173

Figure 7.5.1. Rigid Body with a Flexible Attachment

$$\langle v_x{}^*, \overline{v_x} \rangle = \frac{1}{2} tr\big((B\hat{u})^T B(\hat{\beta}\hat{\Omega} + \hat{\alpha})\big) + \mathbf{a} \cdot \zeta + \frac{1}{2} tr\big((B(\hat{u}\hat{\Omega} + \hat{y}))^T B\hat{\beta}\big)$$

$$+ \mathbf{b} \cdot \eta + \int_0^L \sigma^*(s) \cdot \bar{\sigma}(s)\, ds + \int_0^L \nu^*(s) \cdot \bar{\nu}(s)\, ds,$$

$$(7.5.7)$$

$$= \alpha \cdot u + \mathbf{a} \cdot \zeta + \beta \cdot y + \mathbf{b} \cdot \eta$$

$$+ \int_0^L \sigma^*(s) \cdot \bar{\sigma}(s)\, ds + \int_0^L \nu^*(s) \cdot \bar{\nu}(s)\, ds,$$

where we use the notations $\mathbf{a} = (a_1,\ a_2,\ a_3)$, $\mathbf{b} = (b_1,\ b_2,\ b_3)$, $\zeta = (\zeta_1,\ \zeta_2,\ \zeta_3)$, $\eta = (\eta_1,\ \eta_2,\ \eta_3)$. In the following, we will also adopt the notations $\theta = (\theta_1,\ \theta_2,\ \theta_3)$, $\omega = (\omega_1,\ \omega_2,\ \omega_3)$. The boundary conditions on the configuration space are assumed to be,

$$B^T \sigma(0) = r_0, \qquad (7.5.8a)$$

$$B^T \sigma(L) = r_L, \qquad (7.5.8b)$$

where r_0, r_L are fixed vectors in body coordinates. These boundary conditions on Q give rise to some compatibility conditions on the jet spaces associated with Q. First, we consider the tangent space TQ. The tangent vector $(B\hat{\Omega},\ \omega,\ \nu)$ at $(B,\ \theta,\ \sigma) \in Q$ generates a curve in Q as

174

$$\left(Be^{\epsilon\hat{\Omega}}, \ \theta + \epsilon\omega, \ \sigma + \epsilon\nu\right) \ \in \ Q.$$

Thus the curve must satisfy conditions (7.5.8),

$$(Be^{\epsilon\hat{\Omega}})^T(\sigma + \epsilon\nu)(0) \ = \ r_0, \tag{7.5.9a}$$

$$(Be^{\epsilon\hat{\Omega}})^T(\sigma + \epsilon\nu)(\mathrm{L}) \ = \ r_\mathrm{L}. \tag{7.5.9b}$$

By taking derivatives of both equations with respect to ϵ and then setting ϵ to be 0, we obtain the following compatibility conditions,

$$B^T\nu(0) \ = \ \Omega \times r_0, \tag{7.5.10a}$$

$$B^T\nu(\mathrm{L}) \ = \ \Omega \times r_\mathrm{L}. \tag{7.5.10b}$$

Similarly, for elements in $T_{v_x}TQ$, $\overline{v_x}$ generates a curve in TQ as

$$\left(\ Be^{\epsilon\hat{u}}, \ \theta + \epsilon\zeta, \ \sigma + \epsilon\bar{\sigma}, \ Be^{\epsilon\hat{u}}(\hat{\Omega} + \epsilon\hat{y}), \ \omega + \epsilon\eta, \ \nu + \epsilon\bar{\nu} \ \right). \tag{7.5.11}$$

To satisfy conditions (7.5.8), we need similar conditions for $\bar{\sigma}$ as in (7.5.10),

$$B^T\bar{\sigma}(0) \ = \ u \times r_0, \tag{7.5.12a}$$

$$B^T\bar{\sigma}(\mathrm{L}) \ = \ u \times r_\mathrm{L}. \tag{7.5.12b}$$

From (7.5.10a), we ask,

$$(Be^{\epsilon\hat{u}})^T \ (\nu + \epsilon\bar{\nu})(0) \ = \ (\Omega + \epsilon y) \times r_0,$$

which gives us the condition,

$$B^T\bar{\nu}(0) \ = \ (\hat{u}\hat{\Omega} \ + \ \hat{y}) \ r_0. \tag{7.5.13a}$$

Analogously, from (7.5.10b), we get

$$B^T\bar{\nu}(\mathrm{L}) \ = \ (\hat{u}\hat{\Omega} \ + \ \hat{y}) \ r_\mathrm{L}. \tag{7.5.13b}$$

These compatibility conditions will play an important role in the following derivations.

After the above geometric considerations, we now construct the Lagrangian associated with this system. We assume for the sake of simplification that the inertial reference frame is at the center of mass of the rigid body. This eliminates the kinetic energy of translation of the rigid body. With a similar derivation for the kinetic energy as in Section 4.3, by including the kinetic energy from the string, we write the total kinetic energy as

$$T(v_x) \;=\; \frac{1}{2} < \Omega, \, \mathbf{J}\,\Omega > + \frac{1}{2} < \omega, \mathbf{I}^S \omega > \; + < \Omega, \, \mathbf{I}^S \omega > + \frac{1}{2} \int_0^L \rho(s)|\nu(s)|^2 \, ds, \quad (7.5.14)$$

where $\mathbf{J}$, $\mathbf{I}^S$ are the same as defined in (7.2.2), and ρ is the mass density of the string. We also assume there is no potential energy other than the stored energy in the string which can be written as

$$V(x) \;=\; \frac{1}{2} \int_0^L < AB^T \sigma_s(s), \; B^T \sigma_s(s) > \, ds. \quad (7.5.15)$$

where A is a symmetric stiffness matrix of the string and is assumed to be uniform. The Lagrangian for this system is then,

$$L(v_x) \;=\; T(v_x) \; - \; V(x). \quad (7.5.16)$$

We now generalize the techniques developed in Section 2.2 to the infinite dimensional case. The invariant form of Lagrange-d'Alembert Principle 2.1.9 is still applicable. First, we find the differential of the Lagrangian in (7.5.16). The tangent vector $\overline{v_x} \in T_{v_x} TQ$ generates a curve in TQ as shown in (7.5.11). We compute

$$dL(v_x) \cdot \overline{v_x}$$

$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L(Be^{\epsilon\,\hat{u}}, \; \theta + \epsilon\,\zeta, \; \sigma + \epsilon\,\bar{\sigma}, \; Be^{\epsilon\,\hat{u}}(\hat{\Omega} + \epsilon\,\hat{y}), \; \omega + \epsilon\,\eta, \; \nu + \epsilon\,\bar{\nu}),$$

$$= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \frac{1}{2} < \Omega + \epsilon\,y, \; \mathbf{J}(\Omega + \epsilon\,y) > + \frac{1}{2} < \omega + \epsilon\,\eta, \mathbf{I}^S(\omega + \epsilon\,\eta) >$$

$$+ < \Omega + \epsilon\,y, \; \mathbf{I}^S(\omega + \epsilon\,\eta) > + \frac{1}{2} \int_0^L \rho(s)|\nu(s) + \epsilon\,\bar{\nu}(s)|^2 \, ds$$

$$- \frac{1}{2} \int_0^L < A(Be^{\epsilon\,\hat{u}})^T (\sigma_s + \epsilon\,\bar{\sigma}_s)(s), \; (Be^{\epsilon\,\hat{u}})^T (\sigma_s + \epsilon\,\bar{\sigma}_s)(s) > \, ds.$$

$$= < \mathbf{J}\Omega + \mathbf{I}^S\omega, \; y > + < \mathbf{I}^S(\Omega + \omega), \; \eta > + \int_0^L \rho(s) < \nu(s), \; \bar{\nu}(s) > \, ds$$

$$- \int_0^L < AB^T \sigma_s(s), \; \hat{u}^T B^T \sigma_s(s) > \, ds + \int_0^L < AB^T \sigma_s(s), \; B^T \bar{\sigma}_s(s) > \, ds$$

By using the compatibility conditions (7.5.12), we could further express the differential as

$$dL(v_x) \cdot \overline{v_x} = < \int_0^{\text{L}} \widehat{B^T \sigma_s}(s) AB^T \sigma_s(s)ds - \widehat{r}_{\text{L}} A r_s(\text{L}) + \widehat{r}_0 A r_s(0), \ u >$$

$$+ < \text{J}\Omega + \text{I}^S\omega, \ y > + < \text{I}^S(\Omega + \omega), \ \eta >$$

$$+ \int_0^{\text{L}} < \rho(s)\nu(s), \ \bar{\nu}(s) > ds + \int_0^{\text{L}} < BAB^T \sigma_{ss}(s), \ \bar{\sigma}_s(s) > ds$$

$$(7.5.17)$$

By comparing the above formula with the pairing formula (7.5.7), we could write $dL(v_x)$ in the form of (7.5.6) with the entities,

$$\alpha = \int_0^{\text{L}} \widehat{B^T \sigma_s}(s) AB^T \sigma_s(s)ds - \widehat{r}_{\text{L}} A r_s(\text{L}) + \widehat{r}_0 A r_s(0),$$

$$\mathbf{a} = 0,$$

$$\beta = \text{J}\Omega + \text{I}^S\omega,$$

$$\mathbf{b} = \text{I}^S(\Omega + \omega),$$

$$\sigma^* = BAB^T \sigma_{ss}(s),$$

$$\nu^* = \rho(s)\nu(s).$$

$$(7.5.18)$$

Also we may write the partial derivatives of L as

$$D_1 L(v_x) = \left(B(\hat{\beta}\hat{\Omega} + \hat{\alpha}), \ \mathbf{a}, \ \sigma^*\right), \qquad (7.5.19a)$$

$$D_2 L(v_x) = \left(B\hat{\beta}, \ \mathbf{b}, \ \nu^*\right). \qquad (7.5.19b)$$

Now we could apply an analogous formula as the one in Theorem 2.2.1. Recall that the Lagrange-d'Alembert Principle gives rise to the equations

$$< \frac{d}{dt} D_2 L(v_x), \ z > \ = \ < D_1 L(v_x), \ z > \ + \ < f, \ z >, \quad \forall \ z \in T_x Q, \qquad (7.5.20)$$

where f is the horizontal component of the exterior force which is a horizontal 1-form $\bar{f}$ on TQ. By arguments similar to the ones used in Section 4.3, we determine the representation of the exterior force. First we write the exterior force on the unconstrained tangent bundle TQ_u, cf. (7.5.2),

$$\bar{f} = (B\hat{T}_B, \ S_1\hat{T}_1, \ S_2\hat{T}_2, \ S_3\hat{T}_3, \ \gamma^*, 0, \ 0, \ 0, \ 0, \ 0). \qquad (7.5.21)$$

177

We assume that there are only torques exerted on the connections between the rigid body and the rotors. Thus $\gamma^* := 0$ in (7.5.21). (If there were distributed loads on the flexible attachment, this would not be true any more.) By the characteristics of the torques, we have the condition

$$BT_B = -\sum_{i=1}^{3} S_i T_i,$$

or, from (7.5.3),

$$T_B = -\sum_{i=1}^{3} R(\mathbf{e_i}, \theta_i) T_i.$$

By a similar computation as in (4.3.8), we obtain the form of f as

$$f(v_x) = (0, T^S, 0). \tag{7.5.22}$$

We are ready to find the dynamical equations from (7.5.20). By noting that

$$\dot{B} := B\hat{\Omega}, \quad \dot{\theta} = \omega, \quad \dot{\sigma} = \nu,$$

we have

$$\frac{d}{dt} D_2 L(v_x) = (\dot{B}\hat{\beta} + B\dot{\hat{\beta}}, \dot{\mathbf{b}}, \dot{\nu}^*),$$

$$= (B(\hat{\Omega}\hat{\beta} + \dot{\hat{\beta}}), \dot{\mathbf{b}}, \dot{\nu}^*).$$

Let $z = (B\hat{u}, \zeta, \bar{\sigma})$. Using the pairing formula between T^*Q and TQ, we have

$$< \frac{d}{dt} D_2 L(v_x), z > = \frac{1}{2} tr((B(\hat{\Omega}\hat{\beta} + \dot{\hat{\beta}}))^T B\hat{u}) + < \dot{\mathbf{b}}, \zeta >$$
$$+ \int_0^L < \dot{\nu}^*(s), \bar{\sigma}(s) > ds, \tag{7.5.23}$$

and

$$< D_1 L(v_x), z > = \frac{1}{2} tr((B(\hat{\beta}\hat{\Omega} + \hat{\alpha}))^T B\hat{u}) + < \mathbf{a}, \zeta >$$
$$+ \int_0^L < \sigma^*(s), \bar{\sigma}(s) > ds. \tag{7.5.24}$$

Substituting (7.5.22), (7.5.23), (7.5.24) in (7.5.20), with further simplifications, we obtain the *Principle of Virtual Power*,

$$< \Omega \times \beta + \dot{\beta} - \alpha, u > + < \dot{\mathbf{b}} - \mathbf{a} - T^S, \zeta > + \int_0^L < \dot{\nu}^*(s) - \sigma^*(s), \bar{\sigma}(s) > ds = 0, \tag{7.5.25}$$

178

for all u, ζ, $\bar{\sigma}$, where the elements are given in (7.5.18). Consequently, the classical equations of motion of this system could be written as

$$\dot{\beta} = \beta \times \Omega + \alpha,$$

$$\dot{\mathbf{b}} = \mathbf{a} + T^S,$$

$$\dot{\nu}^* = \sigma^*,$$

By substituting formula (7.5.18) in the above equations, we obtain the dynamical equations for the system of rigid body with string and rotors,

$$\mathbf{J}\dot{\Omega} + \mathbf{I}^S\dot{\omega} = (\mathbf{J}\Omega + \mathbf{I}^S\omega) \times \Omega - \widehat{r_L}Ar_s(\mathbf{L}) + \widehat{r_0}Ar_s(0)$$
$$+ \int_0^L B\widehat{T_{\sigma_s}}(s)AB^T\sigma_s(s)ds,$$

$$\mathbf{I}^S(\dot{\Omega} + \dot{\omega}) = T^S,$$

$$\rho\dot{\nu} = BAB^T\sigma_{ss}, \tag{7.5.26}$$

$$\dot{B} = B\hat{\Omega},$$

$$\dot{\theta} = \omega,$$

$$\dot{\sigma} = \nu.$$

Now we specialize to the situation where there are no rotors on board, which is the case discussed in [38]. Namely, $\mathbf{I}^S = 0$, $T^S = 0$. By defining the *convective variables* associated with the system as

$$r(s, t) = B(t)^T\sigma(s, t), \tag{7.5.27a}$$

$$M(s, t) = \rho(s)B(t)^T\nu(s, t), \tag{7.5.27b}$$

$$m(t) = \mathbf{J}\Omega(t), \tag{7.5.27c}$$

the equations of motion (7.5.26) could be now written as

$$\dot{m} = m \times \mathbf{J}^{-1}m - r_L \times Ar_s(\mathbf{L}) + r_0 \times Ar_s(0) + \int_0^L r_s \times Ar_s ds, \tag{7.5.28a}$$

$$\dot{r} = r \times \mathbf{J}^{-1}m + \frac{1}{\rho} M, \tag{7.5.28b}$$

$$\dot{M} = M \times \mathbf{J}^{-1}m + Ar_{ss}, \tag{7.5.28c}$$

$$\dot{B} = B\widehat{\mathbf{J}^{-1}m}, \tag{7.5.28d}$$

179

with boundary conditions $r(0) = r_0$, and $r(L) = r_L$. It is clear then that the dynamics is decoupled to the so-called *reduced dynamics* (7.5.28a,b,c), and the attitude kinematics (7.5.28d). The reduced dynamics we obtained here are exactly the same dynamical equations derived in [38] by replacing

$$r(0) = a\mathbf{e_2}, \qquad r_s(L) = \mathbf{e_2}.$$

However, the authors adopted a hamiltonian approach in [38].

Now we consider the system in the dual-spin situation. In other words, the rotors are driven to rotate at a constant speed relative to the rigid body. The driven torque T^S are such that $\dot{\omega} = 0$, or

$$T^S = \mathbf{I}^S\dot{\Omega}. \tag{7.5.29}$$

With this feedback law, the equations of motion of the closed loop system can be found from (7.5.26) to be

$$\dot{m} = (m + l) \times \mathbf{J}^{-1}m - r_L \times Ar_s(L) + r_0 \times Ar_s(0) + \int_0^L r_s \times Ar_s ds, \tag{7.5.30a}$$

$$\dot{r} = r \times \mathbf{J}^{-1}m + \frac{1}{\rho}\,M, \tag{7.5.30b}$$

$$\dot{M} = M \times \mathbf{J}^{-1}m + Ar_{ss}, \tag{7.5.30c}$$

$$\dot{B} = B\widehat{\mathbf{J}^{-1}m}, \tag{7.5.30d}$$

where $l = \mathbf{I}^S\omega$ and the convective variables defined in (7.5.27) have been used.

The closed loop system described by (7.5.30) belongs to the category of gyroscopic systems with symmetry. The configuration space is now

$$Q_r = SO(3) \times \mathcal{M}, \tag{7.5.31}$$

with the riemannian metric,

$$\ll (B\hat{u}_1,\ \nu_1),\ (B\hat{u}_2,\ \nu_2) \gg_{(B,\sigma)}\ =\ < u_1,\ \mathbf{J}u_2 >\ +\ \int_0^L \rho(s) < \nu_1(s),\ \nu_2(s) > ds. \tag{7.5.32}$$

The potential energy is given by, cf. (7.5.15),

$$V(B,\ \sigma)\ :=\ \frac{1}{2}\int_0^L < AB^T\sigma_s(s),\ B^T\sigma_s(s) > ds. \tag{7.5.33}$$

180

The gyroscopic field is the same as in the previous examples, cf. (7.3.1),

$$Y(B, \sigma) = (B\widehat{J^{-1}l}, 0). \tag{7.5.34}$$

The Lie group $SO(3)$ acts on the configuration manifold Q_r through the action,

$$\Phi : G \times Q_r \to Q_r,$$
$$R, (B, \sigma) \mapsto (RB, R\sigma). \tag{7.5.35}$$

It is readily seen that the metric, the potential, and the gyroscopic vector field are all G-invariant with respect to this action. The G-invariant Lagrangian can be then written as,

$$L(B, \sigma, B\hat{\Omega}, \nu) = \frac{1}{2} <\Omega, J\Omega> + <\Omega, l> + \frac{1}{2} \int_0^L \rho(s)|\nu(s)|^2 ds$$
$$- \frac{1}{2} \int_0^L <AB^T \sigma_s(s), B^T \sigma_s(s)> ds. \tag{7.5.36}$$

It can be checked that this Lagrangian yields the dynamical equations (7.5.30).

Let $\hat{\xi} \in \mathcal{G} = so(3)$. The infinitesimal generator corresponding to ξ with respect to the action (7.5.35) is

$$\hat{\xi}_{Q_r}(B, \sigma) = (\hat{\xi}B, \hat{\xi}\sigma). \tag{7.5.37}$$

The locked inertia associated with the riemannian metric (7.5.32) can be found through

$$\langle \hat{\xi}, \mathbf{I}_{lock}(B,\sigma)\hat{\eta} \rangle = \ll (\hat{\xi}B, \hat{\xi}\sigma), (\hat{\eta}B, \hat{\eta}\sigma) \gg_{(B,\sigma)},$$
$$= < \xi, \left(BJB^T + \int_0^L \rho(s)\widehat{\sigma(s)}^T \widehat{\sigma(s)} ds \right) \eta >,$$

which gives us

$$\mathbf{I}^o_{lock}(B, \sigma) = BJB^T + \int_0^L \rho(s)\widehat{\sigma(s)}^T \widehat{\sigma(s)} ds. \tag{7.5.38}$$

The gyro-momentum in $\mathcal{G}^*$ corresponding to the gyroscopic field Y defined in (5.4.15) is

$$\mathbf{I}^o_Y(B, \sigma) = Bl. \tag{7.5.39}$$

The induced pre-momentum mapping for this system can be then found to be,

$$\tilde{J}^o(B, \sigma, \zeta) = \left(BJB^T + \int_0^L \rho(s)\widehat{\sigma(s)}^T \widehat{\sigma(s)} ds \right) \zeta + Bl. \tag{7.5.40}$$

Now we study the stability aspects of this problem. With the setup above we could pursue the energy-momentum analysis of relative equilibria as in the previous sections. We do not do this here. Instead, the energy-Casimir method will be used to check the stability conditions for the equilibria of the reduced dynamics (7.5.30a, b, c). We assume that

$$l = l_1 \mathbf{e}_1,$$

$$\mathbf{J} = \text{diag}\{ J_1,\ J_2,\ J_3\ \}, \tag{7.5.41}$$

$$A = \text{diag}\{ A_1,\ A_2,\ A_3\ \}.$$

It can be checked that the point $(m_e,\ r_e,\ M_e)$, where

$$m_e = J_1|\xi|\mathbf{e}_1,$$

$$r_e(s) = a(s)\mathbf{e}_2, \tag{7.5.42}$$

$$M_e(s) = |\xi|\rho(s)a(s)\mathbf{e}_3,$$

gives rise to a relative equilibrium under the condition that the parameters solve the equation

$$A_2 a''(s) + |\xi|^2\rho(s)a(s) = 0, \tag{7.5.43}$$

with the boundary conditions $a(0) = a_0\ (= |r_0|),\ a(L) = a_L\ (= |r_L|)$. Sufficient conditions for stability of this relative equilibrium will be investigated in the following.

Recall that for this system, the energy function on TQ is given by (3.2.12). In terms of the convective variables (m, r, M), the energy can be expressed as

$$
\begin{aligned}
H_L(m, r, M) = {} & \frac{1}{2} < m,\ \mathbf{J}^{-1}m > + \frac{1}{2} \int_0^L \frac{1}{\rho(s)}|M(s)|^2 ds \\
& + \frac{1}{2} \int_0^L < Ar_s(s),\ r_s(s) > ds.
\end{aligned}
\tag{7.5.44}
$$

(This is of course defined on $TQ_r/SO(3)$.) Since the symmetry group is $SO(3)$, one way to construct a Casimir functional is through (5.3.7). Thus from (7.5.40), we obtain a natural Casimir functional for this system expressed in terms of the convective variables,

$$C_\phi(m, r, M) = \frac{1}{2} \phi \left(\left| m + l + \int_0^L r(s) \times M(s)ds \right|^2 \right). \tag{7.5.45}$$

Combining (7.5.43) and (7.5.44), the energy-Casimir functional can be written as

$$H_\phi(m, r, M) = H_L(m, r, M) + C_\phi(m, r, M)$$

$$= \frac{1}{2} < m, \mathbf{J}^{-1}m > + \frac{1}{2} \int_0^L \frac{1}{\rho(s)} M(s)^2 ds$$

$$+ \frac{1}{2} \int_0^L < Ar_s(s), r_s(s) > ds \qquad (7.5.46)$$

$$+ \frac{1}{2} \phi \left(|m + l + \int_0^L r(s) \times M(s) ds|^2 \right).$$

Let

$$\mu = m + l + \int_0^L r(s) \times M(s) ds. \qquad (7.5.47)$$

The first variation of H_ϕ can be found to be

$$DH_\phi(m, r, M) \cdot (\delta m, \delta r, \delta M)$$

$$= < \mathbf{J}^{-1}m, \delta m > + \int_0^L \frac{1}{\rho(s)} < M(s), \delta M(s) > ds$$

$$+ \int_0^L < Ar_s(s), (\delta r)_s(s) > ds + \phi'(|\mu|^2) < \mu, \delta \mu >,$$

$$= < \mathbf{J}^{-1}m + \phi'(|\mu|^2)\mu, \delta m >$$

$$+ \int_0^L < \frac{1}{\rho(s)} M(s) - \phi'(|\mu|^2) r(s) \times \mu, \delta M(s) > ds$$

$$+ \int_0^L < Ar_s(s), (\delta r)_s(s) > ds + \int_0^L < \phi'(|\mu|^2) M(s) \times \mu, \delta r(s) > ds,$$

$$(7.5.48)$$

where

$$\delta \mu = \delta m + \int_0^L \left(\delta r(s) \times M(s) + r(s) \times \delta M(s) \right) ds. \qquad (7.5.49)$$

At relative equilibrium (m_e, r_e, M_e) given by (7.5.42), we have

$$\mu_e = \left((J_1 + \int_0^L \rho(s)a(s)^2 ds) |\xi| + l_1 \right) \mathbf{e_1} = \alpha \, \mathbf{e_1}, \qquad (7.5.50)$$

where we define

$$\alpha \triangleq (J_1 + \int_0^L \rho(s)a(s)^2 ds) |\xi| + l_1. \qquad (7.5.51)$$

Accordingly, in order to have the first variation of H_ϕ vanish at (m_e, r_e, M_e), we impose the following constraint on ϕ,

183

$$\phi'(\alpha^2) = -\frac{|\xi|}{\alpha}. \tag{7.5.52}$$

There is no other requirement on ϕ from the first variation. The second variation of H_ϕ could be computed from (7.5.48). It has the following form,

$$\begin{aligned}
D^2 H_\phi(m, r, M) \cdot (\delta m, \delta r, \delta M) \cdot (\delta m, \delta r, \delta M) \\
= \; <\mathbf{J}^{-1}\delta m, \; \delta m> \; + \; \int_0^L \frac{1}{\rho(s)} <\delta M(s), \; \delta M(s)> ds \\
+ \; 2\phi''(|\mu|^2) <\delta\mu, \; \mu\mu^T\delta\mu> \; + \; \int_0^L <A(\delta r)_s(s), \; (\delta r)_s(s)> ds \\
+ \; \phi'(|\mu|^2) <\delta\mu, \; \delta\mu> \; + \; \phi'(|\mu|^2) <\mu, \; \delta\delta\mu>,
\end{aligned} \tag{7.5.53}$$

where $\delta\delta\mu$ can be derived from (7.5.49),

$$\delta\delta\mu = 2 \int_0^L \delta r(s) \times \delta M(s) ds.$$

At relative equilibrium (m_e, r_e, M_e), we could further write the second variation of H_ϕ as

$$\begin{aligned}
D^2 H_\phi(m_e, r_e, M_e) \cdot (\delta m, \delta r, \delta M) \cdot (\delta m, \delta r, \delta M) \\
= \; <\mathbf{J}^{-1}\delta m, \; \delta m> \; + \; \int_0^L \frac{1}{\rho(s)} | \; \delta M(s) \; | ds \\
+ \; 2\phi''(\alpha^2)\alpha(\delta\mu_1)^2 \; + \; \int_0^L <A(\delta r)_s(s), \; (\delta r)_s(s)> ds \\
- \; \frac{|\xi|}{\alpha} |\delta\mu|^2 \; - \; 2|\xi| \int_0^L \mathbf{e_1} \cdot (\delta r(s) \times \delta M(s)) ds.
\end{aligned} \tag{7.5.54}$$

Also we have, from (7.5.49),

$$\delta\mu = \begin{pmatrix} \delta m_1 \; + \; |\xi| \int_0^L \rho\, a\, \delta r_2\, ds \; + \; \int_0^L a\, \delta M_3\, ds \\ \delta m_2 \; - \; |\xi| \int_0^L \rho\, a\, \delta r_1\, ds \\ \delta m_3 \; - \; \int_0^L a\, \delta M_1\, ds \end{pmatrix}, \tag{7.5.55}$$

where the arguments (s) of the functions in the integrand is assumed. By letting

$$\phi''(\alpha^2) = \frac{|\xi|}{2\alpha^3}, \tag{7.5.56}$$

the second variation could be further written as,

184

$$D^2 H_\phi(m_e, r_e, M_e) \cdot (\delta m, \delta r, \delta M) \cdot (\delta m, \delta r, \delta M)$$

$$= \frac{1}{J_1} \delta m_1^2 + \left(\frac{1}{J_2} - \frac{|\xi|}{\alpha} \right) \delta m_2^2 + \left(\frac{1}{J_3} - \frac{|\xi|}{\alpha} \right) \delta m_3^2$$

$$+ \frac{2|\xi|^2}{\alpha} \delta m_2 \int_0^L \rho \, a \, \delta r_1 \, ds + \frac{2|\xi|}{\alpha} \delta m_3 \int_0^L a \, \delta M_1 \, ds$$

$$- \frac{|\xi|^3}{\alpha} \left(\int_0^L \rho \, a \, \delta r_1 \, ds \right)^2 - \frac{|\xi|}{\alpha} \left(\int_0^L a \, \delta M_1 \, ds \right)^2$$

$$- 2|\xi| \int_0^L \left(\delta r_2 \, \delta M_3 - \delta r_3 \, \delta M_2 \right) ds + \int_0^L \frac{1}{\rho} \left(\delta M_1^2 + \delta M_2^2 + \delta M_3^2 \right) ds$$

$$+ \int_0^L A_1 (\delta r_1)_s^2 + A_2 (\delta r_2)_s^2 + A_3 (\delta r_3)_s^2 \, ds.$$

$$(7.5.57)$$

By the energy-Casimir procedure, we have to check the conditions for positive or negative definiteness of this quadratic form. By comparing (7.5.57) above with Eq. (5.7), p. 88 in [38], we see that they both have exactly the same form. Besides the notations, the only differences are at the moment of inertia and the definition of α. Consequently, we could follow the techniques there to analyze (7.5.57). After nontrivial analysis, we obtain the following sufficient conditions for stability.

$$\alpha - J_2 |\xi| > \frac{|\xi|^3}{A_1 C} \int_0^L \rho(\tau)^2 a(\tau)^2 d\tau, \qquad (7.5.58a)$$

$$\alpha - J_3 |\xi| > \rho(s) \, |\xi| \int_0^L a(\tau)^2 d\tau, \quad \text{for } s \in [0, L], \qquad (7.5.58b)$$

$$\frac{1}{\rho(s)} > \frac{|\xi|^2}{A_2 C}, \quad \text{for } s \in [0, L], \qquad (7.5.58c)$$

$$\frac{1}{\rho(s)} > \frac{|\xi|^2}{A_3 C}, \quad \text{for } s \in [0, L], \qquad (7.5.58d)$$

where C is the lowest eigenvalue of $-d^2/ds^2$ arising in applying Poincaré inequality to simplify (7.5.57), i.e.

$$\int_0^L \left(\frac{df}{ds}(s) \right)^2 ds \geq C \int_0^L f(s)^2 ds. \qquad (7.5.59)$$

for $f(0) = f(L) = 0$. In fact, let $\mathcal{L} = -d^2/ds^2$. Let f, g be in the domain of $\mathcal{L}$ with boundary conditions. We have

$$< \mathcal{L}f, \, g >_{L_2} = < f', \, g' >_{L_2},$$

185

where $< \cdot, \; \cdot >_{L_2}$ denote the L_2 inner product and f' is the first derivative of f. We can express now,

$$\frac{\int_0^L f'(s)^2 ds}{\int_0^L f(s)^2 ds} \; = \; \frac{< \mathcal{L}f, \; f >_{L_2}}{< f, \; f >_{L_2}} \; = \; \Re \; (f),$$

which is the *Rayleigh quotient* associated with the operator $\mathcal{L}$. Since $\mathcal{L}$ is strictly positive and symmetric, from Theorem 5, p. 343 in [67], inf $\Re$ (f) is equal to the lowest eigenvalue of $\mathcal{L}$. By standard calculations, the eigenvalues of $\mathcal{L}$ are $n^2 \pi^2 / L^2$, $n = 1, 2, \cdots$. As a consequence, we have a tight bound in (7.5.59),

$$C \; = \; \frac{\pi^2}{L^2}.$$

By substituting the formula for α, cf. (7.5.51), conditions (7.5.58a) and (7.5.58b) become

$$\left(J_1 - J_2 + \int_0^L \rho(\tau)a(\tau)^2 d\tau \right) |\xi| \; + \; l_1 \; > \; \frac{|\xi|^3}{A_1 C} \int_0^L \rho(\tau)^2 a(\tau)^2 d\tau, \qquad (7.5.60a)$$

$$\left(J_1 - J_3 + \int_0^L \rho(\tau)a(\tau)^2 d\tau \right) |\xi| \; + \; l_1 \; > \; \rho(s) |\xi| \int_0^L a(\tau)^2 d\tau,$$

$$\text{for } s \in [0, L]. \qquad (7.5.60b)$$

It is easy to see that for l_1 sufficiently large, conditions in (7.5.60) are satisfied. We thus only need to check conditions (7.5.58c, d) for stability. We summarize our discussion in the following theorem. The proof automatically follows from the above formulae.

THEOREM 7.5.1

For the system of a rigid body with momentum wheel and attached string, assuming there is only one rotor along a principal axis, and (7.5.41), also assuming the following condition holds,

$$\frac{l_1}{|\xi|} > \max\{ \; \frac{L^2 |\xi|^2}{A_1 \pi^2} \int_0^L \rho(\tau)^2 a(\tau)^2 d\tau \; - \; \left(J_1 - J_2 + \int_0^L \rho(\tau)a(\tau)^2 d\tau \right),$$

$$\rho(s) \int_0^L a(\tau)^2 d\tau \; - \; \left(J_1 - J_3 + \int_0^L \rho(\tau)a(\tau)^2 d\tau \right), \quad \text{for } s \in [0, L] \; \},$$

$$(7.5.61a)$$

the relative equilibrium expressed in terms of convective variables (7.5.42) is formally relatively stable modulo G if

$$|\xi|^2 \;<\; \frac{\pi^2}{L^2 \rho(s)} \;\min\{\, A_2, \; A_3 \,\}, \quad \text{for } s \in [0, L]. \tag{7.5.61b}$$

■

Consequently, no matter which principal axis the rotor spins about, for sufficiently large spinning rate, stability is assured if the assembly rotates at a speed bounded by a material constant. In particular, for a uniform string, $\rho(s) = \rho_0$, we could further write conditions (7.5.61), (7.5.62) as

$$\frac{l_1}{|\xi|} \;>\; \max\{\, \frac{L^2 \rho_0^2 |\xi|^2}{A_1 \pi^2} \int_0^L a(\tau)^2 d\tau - \left(J_1 - J_2 + \rho_0 \int_0^L a(\tau)^2 d\tau \right), \; J_3 - J_1 \,\},$$

$$|\xi|^2 \;<\; \frac{\pi^2}{L^2 \rho_0} \;\min\{\, A_2, \; A_3 \,\}.$$

$$\tag{7.5.62}$$

These conditions (7.5.61), (7.5.62) could be thus used to design a suitable l_1 to ensure stability.

CHAPTER VIII

Conclusions

Some coupled mechanical systems were considered in this dissertation to explore general methodologies in dealing with Eulerian many-body problems arising in spacecraft design. The path of geometry, dynamics and control proved to be helpful in treating these problems. Some of the contributions of this dissertation are outlined in the following.

In Chapter Two, the generalized Euler-Lagrange equations for the special orthogonal group $SO(3)$ were set up so as to make it possible for us to derive dynamical equations for any system including $SO(3)$ as a factor in the configuration space. In particular, this scheme can be easily applied to get the dynamical equations of coupled rigid bodies. Moreover, the developed geometric structures of the group $SO(3)$ help us in handling variational problems on $TSO(3)$ and $T^*SO(3)$. These techniques also were extended to the infinite dimensional case in Chapter Seven in treating string problems.

One of the most important notions introduced in this dissertation is that of a gyroscopic system with symmetry. It generalizes the notion of simple mechanical systems with symmetry to include a term linear with respect to velocity variables in the Lagrangian functional. Such systems are still hamiltonian systems with symmetry. Most of the interesting results for simple mechanical systems with symmetry can be transplanted to gyroscopic systems with symmetry. However, this generalization does enable us to handle a lot more interesting problems such as dual-spin problems, etc. In Chapter Three, the general framework was introduced with some discussions about symmetry, reductions, and relative equilibria. In Chapter Four, we proved that the multibody analog of the dual-spin problem admits as limiting solutions stable relative equilibria of a gyroscopic system with symmetry. In Chapter Five, the beautiful theorem on block diagonalization of the second variation of the energy-momentum functional obtained by Simo, Lewis, Posbergh, and Marsden for simple mechanical systems with

symmetry was extended successfully to the gyroscopic systems with symmetry. The block diagonalization theory and related stability results were used in Chapter Seven on four interesting problems to demonstrate the influence of the gyroscopic field on dynamical behavior, especially, on stability properties. The notion of gyroscopic control was also presented to emphasize the possible role of the gyroscopic field in devising effective control algorithms.

In celestial mechanics, it has been noted that modeling based on point masses is not sufficient to explain observed phenomenon (cf. [75]). It is thus the main theme of Chapter Six to initiate a program of treating the bodies in a gravitational field as bodies of finite extent. For a rigid body in such a force field, we have shown that there are no great circle relative equilibria for asymmetric bodies, contrary to the usual situation for point-mass models. This manifests the difference between the two approaches. In order to account for the problems coming from the integral potential, appropriate approximation schemes were developed to preserve the structure of the system. However, it has been shown that the stability property is not preserved under approximation. Methods introduced in Chapter Five were used to prove a stability result for the approximate models. In contrast to the classical methods, the modern approach here provides systematic and efficient means for solving stability problems.

The research in the future will be continued along these lines. Firstly, the higher order tangent space representations for $SO(3)$ will be generalized to semi-simple Lie groups. Infinite dimensional systems will also be considered. In particular, modeling problems involving tether structures are currently under investigation.

Secondly, properties of the gyroscopic systems with symmetry will be explored in greater detail. In addition to the study of the abstract framework, concrete examples will be further investigated. Problems related to bifurcation, chaos, geometric phases, etc. will be formulated and exploited. Furthermore, the role of damping effects in such systems will be investigated.

Eulerian many-body problems will be further studied. Instead of single rigid body, the problem of coupled rigid and flexible bodies in a gravitational field will be studied.

One interesting example is again the tether problem. We will integrate the techniques developed in this dissertation towards a deeper understanding of tether problems. We expect also that the study of such concrete physical problems will, on the other hand, give us more insight and hints for further development of the general methodologies.

REFERENCES

[1] ABARBANEL, H.D.I. & D.D. HOLM, "Nonlinear Stability Analysis of Three Dimensional Incompressible and Barotropic Fluid," *Phys. Fluids*, 30 (1987), pp. 3369–3382.

[2] ABRAHAM, R. & J.E. MARSDEN, *Foundations of Mechanics,* 2nd ed., Revised, Enlarged, Reset , Benjamin/Cummings, Reading, 1978.

[3] ALVAREZ, G. S., "Geometric Methods of Classical Mechanics Applied to Control Theory," University of California, Berkeley, Ph.D. Dissertation, 1986.

[4] ANTMAN, S.S., *Nonlinear Problems of Elasticity*, in preparation, 1991.

[5] ARNOLD, V.I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[6] ARNOLD, V.I., *Dynamical Systems III*, Encyclopedia of Mathematical Sciences, Vol. 3, Springer-Verlag, Berlin, 1988.

[7] BAILLIEUL, J. & M. LEVI, "Rotational Elastic Dynamics," *Physica D*, 27D (1987), pp. 43–62.

[8] BELETSKII, V.V., *Motion of an Artificial Satellite about its Center of Mass*, Mechanics of Space Flight, Israel Program for Scientific Translations Ltd., 1966.

[9] BERTSEKAS, D.P., *Constrained Optimization and Lagrange Multiplier Methods*, Academic Press, 1982.

[10] BLOCH, A.M., P.S. KRISHNAPRASAD, J.E. MARSDEN & G.Sanchez de ALVAREZ, "Stabilization of Rigid Body Dynamics by Internal and External Torques," 1990, in preparation.

[11] BLOCH, A.M. & J.E. MARSDEN, "Stabilization of Rigid Body Dynamics by the Energy-Casimir Method," *Systems and Control Letters* , 14 (1990), pp. 341–346.

[12] BOTT, R., *Morse Theory and its Applications to Homotory Theory,* Lecture Notes, University of Bonn, 1960.

[13] CHETAYEV, N.G., *The Stability of Motion*, Pergamon Press, 1961.

[14] CROUCH, P.E., "Spacecraft Attitude Control and Stabilization: Applications of Geometric Control Theory to Rigid Body Models," *IEEE Trans. Automatic Control*, 29 (1984), pp. 321–331.

[15] DUBOSHIN, G.N., "On the Problem of Three Rigid Bodies," *Celestial Mechanics*, 33 (1984), pp. 31–47.

[16] ELIPE, R. & A. CID, "On the Motion of Three Bodies: Central Configurations," *Celestial Mechanics*, 37 (1985), pp. 113–126.

[17] ELIPE, R. & S. FERRER, "On the Equilibrium Solutions in the Circular Planar Restricted Three Rigid Bodies Problem," *Celestial Mechanics*, 37 (1985), pp. 59–70.

[18] ERMENKO, E.N., "Stationary Motions in a Restricted Problem of Three Rigid Bodies," *Celestial Mechanics*, 31 (1983), pp. 339–362.

[19] FAN, M.K.H., L.S. WANG, J. KONINCKX & A.L. TITS, "Software Package for Optimization-based Design with User-Supplied Simulators," *IEEE Control Systems Magazine*, 9, 1 (1981), pp. pp. 66–71.

[20] FAN, M.K.H., L.S. WANG, J. KONINCKX & A.L. TITS, "CONSOLE User's Manual," System Research Center, University of Maryland, SRC TR-87-212, College Park, 1987.

[21] FAN, M.K.H., L.S. WANG, J. KONINCKX & A.L. TITS, "CONSOLE: A CAD Tandem for Optimization-Based Design Interacting wth User-Supplied Simulators," *Proc. of the American Control Conf.*, Atlanta, Georgia (June 1988), pp. pp. 701–706.

[22] GANTMACHER, F.R., *The Theory of Matrices*, Volume One, Chelsea Publishing Company, New York, 1960.

[23] GOLDSTEIN, H., *Classical Mechanics*, 2nd Ed., Addison-Wesley Publishing Company, Inc., 1980.

[24] GOTAY, M.J. & J.M. NESTER, "Presymplectic Lagrangian Systems I: the constraint algorithm and the equivalence theorem," *Ann. Inst. Henri Poincaré-Section A*, XXX, 2 (1979), pp. 129–142.

[25] GOTAY, M.J. & J.M. NESTER, "Presymplectic Lagrangian Systems I: the second-order equation problem," *Ann. Inst. Henri Poincaré-Section A*, XXXII, 1(1980), pp. 1–13.

[26] GROBRNER, W., *Lie Series*, B.G. Teubner, Stuttgart, 1963.

[27] GROSSMAN, R., P.S. KRISHNAPRASAD & J.E. MARSDEN, "The Dynamics of Two Coupled Three Dimensional Rigid Bodies," in *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits*, F. SALAM & M. LEVI, eds., SIAM PUBL., Philadelphia, 1988. pp. 373–378 .

[28] HAHN, W., *Stability of Motion*, Springer-Verlag, New York, 1967.

[29] HALE, J.K., *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, 1988.

[30] HAN, J.H., "Symmetries in Nonlinear Control Systems and Their Applications," University of Maryland, College Park, Ph.D. Dissertation, 1986.

[31] HERMANN, R., *Differential Geometry and the Calculus of Variation*, 2nd ed., Math. Sci. Press, 1977.

[32] HIRSCH, M.W. & S. SMALE, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[33] HOLM, D., J.E. MARSDEN, T. RATIU & A. WEINSTEIN, "Nonlinear Stability of Fluid and Plasma Equilibria," *Physics Report*, 123(1985), pp. 1–116.

[34] HUBERT, C., "Dynamics of the Generalized Dual-Spin Turn," *RCA Rev.*, 41(1980), pp. 449–471.

[35] JACOBSON, N., *Lie Algebras*, Dover Publications, Inc., 1962.

[36] KRISHNAPRASAD, P.S., "Lie-Poisson Structures, Dual-spin Spacecraft and Asymptotic Stability," *Nonlinear Analysis: Theory, Methods, and Applications*, 7(1984), pp. 1011–1035.

[37] KRISHNAPRASAD, P.S., "Eulerian Many-Body Problems," Systems Research Center, University of Maryland, College Park, SRC TR-89-15, 1989, also in *Contemp. Math.*, Vol. 97, pp. 187–203, AMS.

[38] KRISHNAPRASAD, P.S. & J.E. MARSDEN, "Hamiltonian Structures and Stability for Rigid Bodies with Flexible Attachments," *Archive for Rational Mechanics and Analysis*, 98 (1987), pp. 71–93.

[39] LANCZOS, C., *The Variational Principles of Mechanics* Fourth Ed., University of Toronto Press, 1970.

[40] LASALLE, J.P., *The Stability of Dynamical Systems*, SIAM, 1976.

[41] LEWIS, D., "Nonlinear Stability of a Rotating Liquid Drop," *Archive for Rational Mechanics and Analysis*, (to appear).

[42] LEWIS, D. & J.C. SIMO, "Nonlinear Stability of Pseudo-rigid Bodies," *Proc. Royal Society*, (to appear).

[43] LIBERMANN, P. & C-M. MARLE, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ., Dordrecht, 1987.

[44] MADDOCKS, J.H., "Stability and Folds," *Archive for Rational Mechanics and Analysis*, 99, 4 (1987), pp. 301–328.

[45] MADDOCKS, J.H., "On the Stability of Relative Equilibria ," to appear in *IMA J. Applied Mathematics* (1990).

[46] MARMO, G., E.J. SALETAN & A. SIMONI, "Reduction of Symplectic Manifolds Through Constants of the Motion," *Nuovo Cimento*, 50B, 1 (1979).

[47] MARSDEN, J.E. & T. RATIU, "Reduction of Poisson Manifolds," *Letters in Math. Phys.*, 11 (1986), pp. 161–169.

[48] MARSDEN, J.E. & A. WEINSTEIN, "Reduction of Symplectic Manifolds with Symmetry," *Reports in Math. Phys.*, 5 (1974), pp. 121–130.

[49] NOMIZU, K., *Lie Groups and Differential Geometry*, The Mathematical Society of Japan, 1956.

[50] OH, Y.G., N. SREENATH, P.S. KRISHNAPRASAD & J.E. MARSDEN, "The Dynamics of Coupled Planar Rigid Bodies Part II: Bifurcation, Periodic Orbits, and Chaos ," *J. Dynamics & Differential Equations*, 1, 3 (1989), pp. 269–298.

[51] PALAIS, R.S., "The Principle of Symmetric Criticality," *Comm. in Math. Physics*, 69, 1(1979), pp. 19–30.

[52] PATRICK, G., "The Dynamics of Two Coupled Rigid Bodies in Three Space," in *Dynamics and Control of Multibody Systems*, J.E. MARSDEN, P.S. KRISHNAPRASAD & J.C. SIMO, eds., AMS, Providence, 1989.

[53] PATRICK, G., "Nonlinear Stability of Coupled Rigid Bodies," Department of Mathematics, University of California, Berkeley, CA, Ph.D. Dissertation, 1990.

[54] POSBERGH, T., "Modeling and Control of Mixed and Flexible Structures ," University of Maryland, College Park, Ph.D. Dissertation, 1988 , also, Systems Research Center Technical Report *SRC TR88-58*.

[55] POSBERGH, T., P.S. KRISHNAPRASAD & J.E. MARSDEN, "Stability Analysis of a Rigid Body with a Flexible Attachment using the Energy-Casimir Method ," in *Differential Geometry : The Interface between Pure and Applied Mathematics*, M. LUKSIC, C. F. MARTIN & W. SHADWICK, eds., AMS , Providence, 1987, also *Contemporary Math.*, 68, AMS, Providence, 253-273. .

[56] POSBERGH, T., J.C. SIMO & J.E. MARSDEN, "Stability Analysis of a Rigid Body with Attached Geometrically Nonlinear Appendage by the Energy-Momentum Method," in *Dynamics and Control of Multibody Systems*, J.E. MARSDEN, P.S. KRISHNAPRASAD & J.C. SIMO, eds., AMS, 1988, pp. 371–397, *Contemporary Mathematics*, Vol. 97.

[57] RAUGEL, G., "Book Review for *Asymptotic Behavior of Dissipative Systems*," *Bulletin of the AMS*, 22, 1(1990), pp. 175–183.

[58] ROUTH, E., *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies* Fourth Ed., Dover Publications, Inc., 1884.

[59] RUMYANTSEV, V.V., "Dynamics and stability of rigid bodies," in *CIME Volume on Stereodynamics*, edizione cremonese, Roma, 1972, pp. 167–271.

[60] SHAHSHAHANI, S., "Dissipative Systems on Manifolds," *Inventiones Mathematicae*, 16(1972), pp. 177–190.

[61] SIMO, J.C., D. LEWIS & J.E. MARSDEN, "Stability of Relative Equilibria, Part I: The Reduced Energy-Momentum Method," Division of Applied Mechanics, Stanford University, SUDAM Report No. 89-3, 1989.

[62] SIMO, J.C., T. POSBERGH & J.E. MARSDEN, "Stability of Relative Equilibria, Part 2: Application to Nonlinear Elasticity," Stanford University, Division of Applied Mechanics., preprint , 1989.

[63] SIMO, J.C., T. POSBERGH & J.E. MARSDEN, "Nonlinear Stability of Geometrically Exact Rods by the Energy-Momentum Method," Stanford University, Division of Applied Mechanics., preprint , 1989.

[64] SMALE, S., "Topology and Mechanics I, II," *Inventiones Mathematicae*, 10, 11 (1970), pp. 305–331, 45–64.

[65] SREENATH, N., "Modeling and Control of Multibody Systems ," University of Maryland, College Park, Ph.D. Dissertation, 1987, also, Systems Research Center Technical Report *SRC TR87-163*.

[66] SREENATH, N., Y.G. OH, P.S. KRISHNAPRASAD & J.E. MARSDEN, "The Dynamics of Coupled Planar Rigid Bodies Part I: Reduction, Equilibria & Stability," *Dynamics & Stability of Systems*, 3, 1&2 (1988), pp. 25–49.

[67] STAKGOLD, I., *Green's Functions and Boundary Value Problems*, John Wiley & Sons, Inc., 1979.

[68] STERNBERG, S., *Lectures on Celestial Mechanics: I and II*, Addison-Wesley, 1969.

[69] VERSHIK, A.M. & L.D. FADDEEV, "Lagrangian Mechanics in Invariant Form," *Sel. Math. Sov.*, 1, 4 (1981), pp. 339–350.

[70] WANG, L.-S. & P.S. KRISHNAPRASAD, "Relative Equilibria of Two Rigid Bodies connected by a Ball-in-Socket Joint," *Proc. of the 1989 IEEE Conference on Decision and Control*, Tampa, FL (Dec. 1989), pp. 692–697.

[71] WANG, L.-S. & P.S. KRISHNAPRASAD, "Multibody Analog of the Dual-Spin Problems," *Proc. of the 1989 IEEE Conference on Decision and Control*, to appear (Dec. 1990).

[72] WANG, L.-S., P.S. KRISHNAPRASAD & J. MADDOCKS, "Hamiltonian Dynamics of a Rigid Body in a Central Gravitational Field," Systems Research Center, University of Maryland, College Park, TR-90-3, 1990.

[73] WHITTAKER, E.T., *A Treatise on the Analytical Dynamics of Particals and Rigid Bodies,* 4th Ed., Cambridge University Press, Cambridge, 1959.

[74] WILLMORE, T.J., "The Definition of Lie Derivative," *Proc. Edin. Math. Soc.,* 12, 2 (1960), pp. 27–29.

[75] WISDOM, J., "Rotational Dynamics of Irregularly Shaped Natural Satellites," *The Astronomical Journal,* 94, 5 (1987), pp. 1350–1360.

[76] WITTENBURG, J., *Dynamics of Multibody Systems,* B.G. Teubner, Stuttgart, 1977.